

# On the Signature of the Shapovalov Form

by

Wai Ling Yee

Bachelor of Mathematics, University of Waterloo, 2000.

Submitted to the Department of Mathematics  
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June 2004

© Wai Ling Yee, MMIV. All rights reserved.

The author hereby grants to MIT permission to reproduce and distribute publicly paper  
and electronic copies of this thesis document in whole or in part.

Author .....  
Department of Mathematics  
April 30, 2004

Certified by .....  
David A. Vogan, Jr.  
Professor of Mathematics  
Thesis Supervisor

Accepted by .....  
Pavel Etingof  
Chairman, Department Committee on Graduate Students



# On the Signature of the Shapovalov Form

by

Wai Ling Yee

Submitted to the Department of Mathematics  
on April 30, 2004, in partial fulfillment of the  
requirements for the degree of  
Doctor of Philosophy

## Abstract

Classifying the irreducible unitary representations of a real reductive group is equivalent to the algebraic problem of classifying the Harish-Chandra modules admitting a positive definite invariant Hermitian form. Finding a formula for the signature of the Shapovalov form is a related problem which may be a necessary first step in such a classification.

A Verma module may admit an invariant Hermitian form, which is unique up to multiplication by a real scalar when it exists. Suitably normalized, it is known as the Shapovalov form. The collection of highest weights decomposes under the affine Weyl group action into alcoves. The signature of the Shapovalov form for an irreducible Verma module depends only on the alcove in which the highest weight lies. We develop a formula for this signature, depending on the combinatorial structure of the affine Weyl group.

Thesis Supervisor: David A. Vogan, Jr.

Title: Professor of Mathematics



## Acknowledgements

First and foremost, I would like to thank my advisor, Professor Vogan, who surely must be the world's most patient man. I could not have hoped for better guidance or for a more helpful, knowledgeable, and understanding advisor.

Among the many people who were mathematically influential for the duration of this degree, I am particularly indebted to Alberto DeSole, for his help and for being an affable mathematical guinea pig; to Anthony Henderson, who not only offered help but was a good source of advice; to Kevin McGerty, for his endless patience; and to Professor Hare, for her encouragement.

I would like to thank Linda Okun and Dennis Porche for their assistance in administrative matters and also for adding much warmth to the department.

I would also like to thank my family, especially Min and Rob for their moral support.

Over the years that I've spent at MIT, many people have come to mean very much to me. I feel particularly fortunate to have become friends with Tilman, Cilanne, Etienne, Benoit, Yvonne, the time sinks Dion and Reejis, Brian, and Karen. Thanks for the laughs we shared, for helping me keep my life balanced, for expanding my horizons, and for all the other little things which helped me maintain my sanity and morale. Thanks also to officemates Kari, Thomas, Zhou; hockey buddies Peter, Francis, Todd, and the Eulers; musicians Andrew, Keala, Justin, and Fred; Jan, Ian, Mike, and the inhabitants of 26 Calvin Street.

While at MIT, I have been reminded constantly of the value of old friends and familiar faces. I would like to thank Ming for his constant encouragement. Friendships with Jason, Sam, Richard, Steve, Keith, Suresh, Cyrus, Dave, Ondřej, Fred, and Brenda have been greatly appreciated.

This thesis was partially supported by NSERC postgraduate scholarships and by the NSF.



# Contents

<b>1</b>	<b>Introduction</b>	<b>11</b>
1.1	Unitarizability and invariant Hermitian forms . . . . .	11
1.2	Historical background . . . . .	12
<b>2</b>	<b>An introduction to the Shapovalov form</b>	<b>15</b>
<b>3</b>	<b>The Jantzen filtration</b>	<b>23</b>
<b>4</b>	<b>A preliminary formula for the signature character</b>	<b>31</b>
<b>5</b>	<b>Calculating <math>\varepsilon</math></b>	<b>35</b>
5.1	Dependence on Weyl chambers . . . . .	36
5.2	Calculating $\varepsilon$ for the rank 2 case . . . . .	40
5.3	Using induction to obtain the general case . . . . .	54
<b>6</b>	<b>Extending to non-compact <math>\mathfrak{h}</math></b>	<b>63</b>
<b>7</b>	<b>Historical context</b>	<b>71</b>
<b>8</b>	<b>Conclusion</b>	<b>77</b>





# List of Figures

5-1	Classical rank 2 systems . . . . .	36
5-2	Some examples . . . . .	38
5-3	Type $A_2$ . . . . .	42
5-4	Type $A_2$ : calculating $\varepsilon$ for $H_{\alpha_1+\alpha_2,N}$ . . . . .	56
5-5	Type $B_2$ : calculating $\varepsilon$ for $H_{\alpha_1+\alpha_2,N}$ . . . . .	58
5-6	Type $B_2$ : calculating $\varepsilon$ for $H_{\alpha_1+2\alpha_2,N}$ . . . . .	59



# Chapter 1

## Introduction

### 1.1 Unitarizability and invariant Hermitian forms

Classically, the fundamental concept of Fourier analysis was that an essentially arbitrary function could be expanded as a linear combination of exponentials. The more recent development of ideas in group theory has illuminated the dependence of results in Fourier analysis on group-theoretic concepts, resulting in the movement from Euclidean spaces to the more general setting of locally compact groups. Results such as the Peter-Weyl Theorem give us a means of decomposing function spaces of a compact group  $G$  into an orthogonal direct sum of subspaces expressed in terms of characters of irreducible unitary representations of  $G$ . Equipped with this decomposition and knowledge of these simpler subspaces, one can reformulate problems in analysis in more tractable settings. Quantum mechanics is another source of problems connected to unitary representations. Because of its implications for many different areas of mathematics and physics, the study of unitary representations has been an active area of research.

The irreducible unitary representations of an abelian group are one dimensional (characters). In the case of a locally compact abelian group, Pontrjagin showed that the unitary dual (the set of equivalence classes of irreducible unitary representations)  $\widehat{G}_u$ , furnished with pointwise multiplication of characters as the product, has the structure of a locally compact abelian group. In this situation, the unitary dual has the additional property that its unitary dual is  $G$ .

Investigation of the non-abelian case began with the study of compact groups. In the 1920s, Weyl described the irreducible unitary representations of a compact, connected Lie group. For a

locally compact group, (for example, a real or complex reductive group), the problem of describing the unitary dual remains unsolved, with the exception of some special cases.

In the interests of classifying the irreducible unitary representations, we wish to study a broader family of representations: those which admit an invariant Hermitian form. Unitarity of a representation amounts to the existence of a positive definite invariant Hermitian form on the underlying vector space, hence our objective will be, in particular, to investigate the signatures of invariant Hermitian forms and to understand how positivity can fail.

## 1.2 Historical background

Let  $G$  be a real reductive Lie group and  $K$  be a maximal compact subgroup of  $G$ . Let  $\mathfrak{g}_0$  and  $\mathfrak{k}_0$  be the corresponding Lie algebras, and let  $\mathfrak{g}$  and  $\mathfrak{k}$  be their complexifications. A Harish-Chandra module  $M$  is a complex vector space which is:

a) a  $(\mathfrak{g}, K)$ -module:

$M$  has compatible actions by  $\mathfrak{g}$  and  $K$ , and every  $m \in M$  lies in a finite-dimensional  $K$ -invariant subspace

b) admissible:

the  $i$ -isotypic subspace of  $M$  is finite-dimensional for every irreducible unitary representation  $i$  of  $K$

c) finitely generated over  $U(\mathfrak{g})$ .

To an admissible representation  $(\pi, V)$  of  $G$ , we associate in a natural way a Harish-Chandra module  $V_{K-\text{finite}}$ , known as *the Harish-Chandra module of  $V$* . We define  $V_{K-\text{finite}}$ , the set of  $K$ -finite vectors, to be the set of vectors which lie in a finite dimensional  $K$ -invariant subspace of  $V$ .

For irreducible unitary representations, infinitesimal equivalence (the Harish-Chandra modules are isomorphic) implies unitary equivalence. Furthermore, for any irreducible Harish-Chandra module  $M$  with a positive definite invariant Hermitian form, one can construct an irreducible unitary representation  $(\pi, V)$  so that  $M$  is the Harish-Chandra module of  $V$ . (See [11].) It follows that classifying the irreducible unitary representations of  $G$  is equivalent to the algebraic problem of classifying the Harish-Chandra modules admitting a positive definite invariant Hermitian form.

Verma modules may admit an invariant Hermitian form, which is unique up to multiplication by a real scalar when it exists. Suitably normalized, this Hermitian form is called the Shapovalov form. Finding a formula for the signature of the Shapovalov form is a related problem which may be a necessary first step in classifying the Harish-Chandra modules admitting a positive definite invariant Hermitian form. The Shapovalov form on  $M(\lambda)$  exists for  $\lambda$  in a subspace of  $\mathfrak{h}^*$ , where  $\mathfrak{h}$  is a maximally compact Cartan subalgebra. This will be discussed further in Chapter 2. Previously, Nolan Wallach computed the signature of the Shapovalov form for a region corresponding roughly to the intersection of that subspace with the negative Weyl chamber. In the following, we will describe its implications for unitarizability of  $(\mathfrak{g}, K)$ -modules.

In lectures at the Institute for Advanced Studies in 1978, Zuckerman introduced an algebraic method of constructing all admissible  $(\mathfrak{g}, K)$ -modules using homological algebra machinery known as cohomological induction. (See [8].)

Let  $L$  be a  $\theta$ -stable Levi subgroup of  $G$  with corresponding complexified Lie algebra  $\mathfrak{l}$ , and let  $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$  be a parabolic subalgebra of  $\mathfrak{g}$ . Observe that representations of  $\mathfrak{l}$  can be extended to representations of  $\mathfrak{q}$  by allowing  $\mathfrak{u}$  to act trivially.

Let  $\mathcal{C}(\mathfrak{g}, \mathfrak{k})$  be the category of  $(\mathfrak{g}, \mathfrak{k})$ -modules. Consider the induction functor

$$\mathrm{ind}_{\mathfrak{q}, \mathfrak{l} \cap \mathfrak{k}}^{\mathfrak{g}, \mathfrak{l} \cap \mathfrak{k}}(Z) = U(\mathfrak{g}) \otimes_{U(\mathfrak{q})} Z$$

from  $\mathcal{C}(\mathfrak{q}, \mathfrak{l} \cap \mathfrak{k})$  to  $\mathcal{C}(\mathfrak{g}, \mathfrak{l} \cap \mathfrak{k})$ . The induction functor, when applied to  $Z = \mathbb{C}_\lambda \otimes V$  where  $\lambda \in z(\mathfrak{l})^*$  and  $V$  is an  $(\mathfrak{l}, L \cap K)$ -module, produces what are known as generalized Verma modules. When applied to  $Z = \mathbb{C}_\lambda$  in the special case where our parabolic subalgebra is a Borel subalgebra, it produces the Verma module of highest weight  $\lambda$ .

Let the functor  $\Gamma : \mathcal{C}(\mathfrak{g}, \mathfrak{l} \cap \mathfrak{k}) \rightarrow \mathcal{C}(\mathfrak{g}, \mathfrak{k})$  be such that  $\Gamma(V)$  is the set of  $\mathfrak{k}$ -finite vectors of  $V$ . The functor  $\Gamma$  is covariant and left exact. As  $\mathcal{C}(\mathfrak{g}, \mathfrak{l} \cap \mathfrak{k})$  has enough injectives, we can form the Zuckerman functors:  $\Gamma^j$  = the  $j^{\mathrm{th}}$  derived functor of  $\Gamma$ .

By composing the induction functor with the Zuckerman functors  $\Gamma^j : \mathcal{C}(\mathfrak{g}, \mathfrak{l} \cap \mathfrak{k}) \rightarrow \mathcal{C}(\mathfrak{g}, \mathfrak{k})$ , we obtain cohomological induction functors which take  $(\mathfrak{l}, \mathfrak{l} \cap \mathfrak{k})$ -modules to  $(\mathfrak{g}, \mathfrak{k})$ -modules.

In [2], Enright and Wallach show for admissible  $V \in \mathcal{C}(\mathfrak{g}, \mathfrak{l} \cap \mathfrak{k})$  and  $m$  equal to the dimension of the compact part of  $\mathfrak{u}$  that  $\Gamma^j(V^h) \simeq (\Gamma^{2m-j}(V))^h$ , where the superscript  $h$  denotes Hermitian dual. In particular, if  $V$  admits a non-degenerate invariant Hermitian form so that  $V^h \simeq V$ , then  $\Gamma^m(V) \simeq (\Gamma^m(V))^h$ . Thus  $\Gamma^m(V)$  also admits a non-degenerate invariant Hermitian form.

Subsequently in [13], Wallach lifts information concerning the signature of the invariant Hermitian form on  $V \in \mathcal{C}(\mathfrak{l}, \mathfrak{l} \cap \mathfrak{k})$  to the invariant Hermitian form (known as the *Shapovalov form*, which we will describe further in the following section) on the generalized Verma module  $\text{ind}_{\mathfrak{q}, \mathfrak{l} \cap \mathfrak{k}}^{\mathfrak{g}, \mathfrak{l} \cap \mathfrak{k}}(\mathbb{C}_\lambda \otimes V)$ . Finally, he lifts that information, using knowledge of the isomorphism  $\Gamma^m(X) \simeq (\Gamma^m(X))^h$ , to the form on the cohomologically induced  $(\mathfrak{g}, \mathfrak{k})$ -module  $\Gamma^m \left( \text{ind}_{\mathfrak{q}, \mathfrak{l} \cap \mathfrak{k}}^{\mathfrak{g}, \mathfrak{l} \cap \mathfrak{k}}(\mathbb{C}_\lambda \otimes V) \right)$ . He concludes that if the form on  $V$  is positive definite and  $\lambda$  lies in a particular region bounded by hyperplanes, which we shall call the Wallach region, then the  $(\mathfrak{g}, \mathfrak{k})$ -module produced is also unitarizable.

In this thesis, we will extend the formula for the signature of the Shapovalov form beyond the Wallach region. We will compute the signature of the Shapovalov form on all irreducible Verma modules which admit an invariant Hermitian form.

## Chapter 2

# An introduction to the Shapovalov form

We will use the following notation:

- $\mathfrak{g}_0$  denotes a real semisimple Lie algebra
- $\theta$  is a Cartan involution of  $\mathfrak{g}_0$
- $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$  is the Cartan decomposition corresponding to  $\theta$
- $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$  is a  $\theta$  stable Cartan subalgebra and corresponding Cartan decomposition

We drop the subscript 0 to denote complexification. We let  $B(\cdot, \cdot)$  denote the Killing form, which is a symmetric, invariant, non-degenerate bilinear form on  $\mathfrak{g}$ . We let  $(\cdot, \cdot)$  denote the symmetric bilinear form on  $\mathfrak{h}^*$  induced by  $B$ .

**Definition 2.1.** A Hermitian form  $\langle \cdot, \cdot \rangle$  on a  $\mathfrak{g}$ -module  $V$  is **invariant** if it satisfies

$$\langle Xv, w \rangle + \langle v, \bar{X}w \rangle = 0$$

for every  $X \in \mathfrak{g}$  and every  $v, w \in V$ , where  $\bar{X}$  denotes the complex conjugate of  $X$  with respect to the real form  $\mathfrak{g}_0$ .

We wish to define the Hermitian dual of a representation of  $\mathfrak{g}$ . In order to do so, we first define the conjugate representation:

**Definition 2.2.** Given a representation  $(\pi, V)$ , we define the **conjugate representation**  $(\bar{\pi}, \bar{V})$  as follows: the vector space  $\bar{V}$  is the same vector space as  $V$ , but with the following definition of multiplication by a complex scalar:  $z\bar{v} = \bar{z} \cdot v$  where by  $\cdot$  and  $\bar{\cdot}$  we mean scalar multiplication in  $V$  and  $\bar{V}$ , respectively. We define  $\bar{\pi}(X) = \pi(\bar{X})$  for all  $X \in \mathfrak{g}$ , where conjugation is with respect to the real form  $\mathfrak{g}_0$ .

Observe that every weight  $\mu$  of  $V$  under  $\pi$  gives rise to a weight  $\bar{\mu}$  of  $\bar{V}$  under  $\bar{\pi}$ , where  $\bar{\mu}(H) = \overline{\mu(\bar{H})}$  for every  $H \in \mathfrak{h}$ .

**Definition 2.3.** The **Hermitian dual** of the representation  $(\pi, V)$  is  $(\pi^h, V^h)$ , the conjugate representation of the contragredient representation of  $(\pi, V)$ .

If  $V$  is the direct sum of weight spaces  $V_\mu$  for  $\mu \in I$ , then  $V^h$  is the direct product of weight spaces  $V_{-\bar{\mu}}^h$  for  $\mu \in I$ .

**Theorem 2.4.** An irreducible representation  $(\pi, V)$  admits a non-degenerate invariant Hermitian form if and only if  $(\pi, V)$  is isomorphic to a subrepresentation of its Hermitian dual.

**Definition 2.5.** In the case where  $(\pi, V)$  is the Verma module  $M(\lambda)$  with generator  $v_\lambda$ , the **Shapovalov form**, which we will denote by  $\langle \cdot, \cdot \rangle_\lambda$ , is the invariant Hermitian form for which  $\langle v_\lambda, v_\lambda \rangle_\lambda = 1$ .

According to the previous theorem, in order to determine when the Shapovalov form exists, we wish to determine when a Verma module embeds in its Hermitian dual.

Pick some positive system of roots  $\Delta^+(\mathfrak{g}, \mathfrak{h})$  and let  $\mathfrak{b}$  be the corresponding Borel subalgebra and  $\mathfrak{n}$  its nilradical. The production functor is defined by  $\text{pro}_{\mathfrak{b}}^{\mathfrak{g}}(V) = \text{Hom}_{\mathfrak{b}}(U(\mathfrak{g}), V)$ , where  $V$  is a  $\mathfrak{b}$ -module. We have  $\text{ind}_{\mathfrak{b}}^{\mathfrak{g}}(V)^h \simeq \text{pro}_{\mathfrak{b}}^{\mathfrak{g}}(V^h)$  (Lemma 5.13, [12]). We conclude that the Hermitian dual of the Verma module  $M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda = \text{ind}_{\mathfrak{b}}^{\mathfrak{g}}(\mathbb{C}_\lambda)$  is  $\text{pro}_{\mathfrak{b}}^{\mathfrak{g}}(\mathbb{C}_\lambda^h) = \text{Hom}_{\bar{\mathfrak{b}}}(U(\mathfrak{g}), \mathbb{C}_{-\bar{\lambda}})$ . Now  $\text{Hom}_{\bar{\mathfrak{b}}}(U(\mathfrak{g}), \mathbb{C}_{-\bar{\lambda}})$  has the same weights as  $U(\mathfrak{g}) \otimes_{\bar{\mathfrak{b}}^{op}} \mathbb{C}_{-\bar{\lambda}}$ . We conclude from universality properties of Verma modules that the Verma module  $U(\mathfrak{g}) \otimes_{\bar{\mathfrak{b}}^{op}} \mathbb{C}_{-\bar{\lambda}}$  embeds into the Hermitian dual of  $M(\lambda)$ . From this, we conclude that  $M(\lambda)$  admits an invariant Hermitian form if  $-\bar{\lambda} = \lambda$  and  $\overline{\Delta^+(\mathfrak{g}, \mathfrak{h})} = \Delta^-(\mathfrak{g}, \mathfrak{h})$ . Observe that we must have  $\mathfrak{b} \cap \bar{\mathfrak{b}} = \mathfrak{h}$ . In the following, we will determine for which  $\mathfrak{h}$  and  $\lambda$  these conditions are satisfied.

Assume that  $\mathfrak{h}_0$  is  $\theta$ -stable. For a  $\theta$ -stable Cartan subalgebra  $\mathfrak{h}_0$  of  $\mathfrak{g}_0$  with Cartan decomposition  $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$ , a root  $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$  is imaginary valued on  $\mathfrak{t}_0$  and real valued on  $\mathfrak{a}_0$ . A root  $\alpha$  is



**imaginary** if it vanishes on  $\mathfrak{a}_0$  and **real** if it vanishes on  $\mathfrak{t}_0$ . If  $\alpha$  has support on both  $\mathfrak{t}_0$  and  $\mathfrak{a}_0$ , then it is **complex**.

We define  $\theta\alpha$  by  $(\theta\alpha)(H) = \alpha(\theta^{-1}H)$  for every  $H \in \mathfrak{h}$ . If  $X_\alpha \in \mathfrak{g}_\alpha$ , then

$$[H, \theta X_\alpha] = \theta([\theta^{-1}H, X_\alpha]) = \alpha(\theta^{-1}H)\theta X_\alpha = (\theta\alpha)(H)\theta X_\alpha.$$

Therefore if  $\alpha$  is a root, then  $\theta\alpha$  is a root. We have  $\theta\mathfrak{g}_\alpha = \mathfrak{g}_{\theta\alpha}$ .

We define  $\bar{\alpha}$  by  $\bar{\alpha}(H) = \overline{\alpha(\bar{H})}$  for every  $H \in \mathfrak{h}$ . As  $\bar{\cdot}$  is involutive and  $[\bar{X}, \bar{Y}] = \overline{[X, Y]}$ , arguing as for  $\theta$ , we conclude that  $\bar{\alpha}$  is a root if  $\alpha$  is a root. Also,  $\bar{\mathfrak{g}}_\alpha = \mathfrak{g}_{\bar{\alpha}}$ . Note that  $\bar{\alpha} = \alpha$  if and only if  $\alpha$  is real, and  $\bar{\alpha} = -\alpha$  if and only if  $\alpha$  is imaginary.

In fact,  $\theta\alpha$  and  $\bar{\alpha}$  are related by  $\theta\alpha = -\bar{\alpha}$  as  $\alpha$  is imaginary valued on  $\mathfrak{t}_0$  and real valued on  $\mathfrak{a}_0$ .

Since  $\theta\alpha = \alpha$  for imaginary  $\alpha$ , therefore  $\theta\mathfrak{g}_\alpha = \mathfrak{g}_\alpha$  and so we have  $\mathfrak{g}_\alpha = \mathfrak{g}_\alpha \cap \mathfrak{k} \oplus \mathfrak{g}_\alpha \cap \mathfrak{p}$ . As  $\mathfrak{g}_\alpha$  is one-dimensional, therefore  $\mathfrak{g}_\alpha = \mathfrak{g}_\alpha \cap \mathfrak{k}$  or  $\mathfrak{g}_\alpha = \mathfrak{g}_\alpha \cap \mathfrak{p}$ . We call an *imaginary* root  $\alpha$  **compact** if  $\mathfrak{g}_\alpha \subset \mathfrak{k}$  and **noncompact** if  $\mathfrak{g}_\alpha \subset \mathfrak{p}$ .

We define  $B_\theta(\cdot, \cdot) = -B(\cdot, \theta\cdot)$ . As  $B$  is symmetric and invariant and  $\theta$  is an involutive automorphism of  $\mathfrak{g}$ ,  $B_\theta$  is symmetric. Since  $\mathfrak{k}$  and  $\mathfrak{p}$  are the eigenspaces corresponding to the eigenvalues 1 and  $-1$  of  $\theta$ , respectively, we conclude that  $\mathfrak{k}$  and  $\mathfrak{p}$  are orthogonal with respect to  $B$ . The decomposition  $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$  is both direct and orthogonal, hence  $\mathfrak{h}^* = \mathfrak{t}^* \oplus \mathfrak{a}^*$  is an orthogonal decomposition of  $\mathfrak{h}^*$  with respect to the non-degenerate symmetric bilinear form induced by  $B$ . For every  $\alpha \in \mathfrak{h}^*$ , we let  $\alpha = \alpha_{\mathfrak{t}} + \alpha_{\mathfrak{a}}$  be the decomposition of  $\alpha$  under this direct sum. Note that  $\alpha|_{\mathfrak{t}} = \alpha_{\mathfrak{t}}|_{\mathfrak{t}}$ ,  $\alpha|_{\mathfrak{a}} = \alpha_{\mathfrak{a}}|_{\mathfrak{a}}$ , and  $\alpha_{\mathfrak{t}}$  and  $\alpha_{\mathfrak{a}}$  are orthogonal.

A Cartan subalgebra  $\mathfrak{h}$  is **maximally compact** or **fundamental** if the compact part has largest possible dimension. In this case, there are no real roots, whence every root has non-trivial restriction to  $\mathfrak{t}$  (see Proposition 6.70 of [7]). Suppose  $\mathfrak{h}$  is maximally compact. If  $X_\alpha \in \mathfrak{g}_\alpha$  where  $\alpha$  is complex, then  $\theta\alpha = \alpha_{\mathfrak{t}} - \alpha_{\mathfrak{a}}$  and  $\alpha$  have the same restriction to  $\mathfrak{t}$ . The vectors  $X_\alpha + \theta X_\alpha \in \mathfrak{k}$  and  $X_\alpha - \theta X_\alpha \in \mathfrak{p}$  both have  $\mathfrak{t}$ -weight  $\alpha_{\mathfrak{t}}$ . If  $\alpha \in \Delta(\mathfrak{k}, \mathfrak{t})$  arises from the imaginary root  $\beta \in \Delta(\mathfrak{g}, \mathfrak{h})$ , then  $\beta$  is the only root restricting to  $\alpha$ . If  $\alpha$  arises from a complex root  $\beta$ , then  $\beta$  and  $\theta\beta$  are the only roots restricting to  $\alpha$ . We may think of  $\Delta(\mathfrak{g}, \mathfrak{h})$  as  $\Delta(\mathfrak{k}, \mathfrak{t}) \sqcup \Delta(\mathfrak{p}, \mathfrak{t})$ , where  $\Delta(\mathfrak{k}, \mathfrak{t})$  and  $\Delta(\mathfrak{p}, \mathfrak{t})$  overlap in the part coming from complex roots. Therefore we may think of the compact roots as  $\Delta(\mathfrak{k}, \mathfrak{t})$  and the noncompact roots as  $\Delta(\mathfrak{p}, \mathfrak{t})$ .

**Claim 2.6.** *We have  $\overline{\Delta^+(\mathfrak{g}, \mathfrak{h})} = \Delta^-(\mathfrak{g}, \mathfrak{h})$  for some appropriate choice of  $\Delta^+(\mathfrak{g}, \mathfrak{h})$  if and only if*

every  $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$  has non-trivial restriction to  $\mathfrak{t}$  (i.e.  $\mathfrak{h}$  is maximally compact).

*Proof.*  $\Rightarrow$ : This direction is clear as we cannot have  $\bar{\alpha} = \alpha$ , and so none of the roots are real.

$\Leftarrow$ : Conversely, if  $\mathfrak{h}$  is maximally compact, then  $\mathfrak{t}$  is a Cartan subalgebra of  $\mathfrak{k}$ . We know that  $\mathfrak{k}$  is a reductive Lie subalgebra and every  $\alpha \in \Delta(\mathfrak{k}, \mathfrak{t})$  is the restriction of some  $\beta \in \Delta(\mathfrak{g}, \mathfrak{h})$  to  $\mathfrak{t}$ . Choose a positive system  $\Delta^+(\mathfrak{k}, \mathfrak{t})$  for  $\Delta(\mathfrak{k}, \mathfrak{t})$  defined by some regular element  $r_{\mathfrak{k}} \in \mathfrak{t}^*$ . We can arrange for  $r_{\mathfrak{k}}$  to be regular with respect to the root system  $\Delta(\mathfrak{g}, \mathfrak{h})$  also as every  $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$  has non-zero restriction to  $\mathfrak{t}$ . We define  $\Delta^+(\mathfrak{g}, \mathfrak{h})$  to be the positive system of  $\Delta(\mathfrak{g}, \mathfrak{h})$  corresponding to  $r_{\mathfrak{k}}$ . Since  $(\alpha, r_{\mathfrak{k}}) = (\alpha|_{\mathfrak{t}}, r_{\mathfrak{k}})$ , we conclude that  $\Delta^+(\mathfrak{g}, \mathfrak{h})$  is compatible with  $\Delta^+(\mathfrak{k}, \mathfrak{t})$ : if  $\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{h})$  and  $\alpha|_{\mathfrak{t}} \in \Delta(\mathfrak{k}, \mathfrak{t})$ , then  $\alpha|_{\mathfrak{t}} \in \Delta^+(\mathfrak{k}, \mathfrak{t})$ . Furthermore, as  $\bar{\alpha} = -\alpha_{\mathfrak{t}} + \alpha_{\mathfrak{a}}$ , we see that we have  $\overline{\Delta^+(\mathfrak{g}, \mathfrak{h})} = \Delta^-(\mathfrak{g}, \mathfrak{h})$ .  $\square$

**Remark 2.7.** We may also write the condition  $\overline{\Delta^+(\mathfrak{g}, \mathfrak{h})} = \Delta^-(\mathfrak{g}, \mathfrak{h})$  as  $\theta\Delta^+(\mathfrak{g}, \mathfrak{h}) = \Delta^+(\mathfrak{g}, \mathfrak{h})$ .

We may satisfy the final condition by selecting  $\lambda$  to be imaginary—that is, it takes imaginary values on  $\mathfrak{t}_0 \oplus \mathfrak{a}_0$ . In conclusion,

**Proposition 2.8.** Let  $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$  be a Borel subalgebra of  $\mathfrak{g}$ . If  $\mathfrak{h} = \mathfrak{b} \cap \bar{\mathfrak{b}}$ ,  $\mathfrak{h}$  is maximally compact,  $\lambda$  is imaginary, and the positive system  $\Delta^+(\mathfrak{g}, \mathfrak{h})$  corresponding to  $\mathfrak{b}$  is  $\theta$ -stable, then the Verma module  $M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_{\lambda}$  admits an invariant Hermitian form.

In this case, how does one construct the Shapovalov form?

For  $X \in \mathfrak{g}$ , let  $X^* = -\bar{X}$  and extend the map  $X \mapsto X^*$  to an involutive antiautomorphism of  $U(\mathfrak{g})$  by  $1^* = 1$  and  $(xy)^* = y^*x^*$  for every  $x, y \in U(\mathfrak{g})$ . We have  $U(\mathfrak{g}) = U(\mathfrak{h}) \oplus (U(\mathfrak{g})\mathfrak{n} + \mathfrak{n}^{op}U(\mathfrak{g}))$  from the triangular decomposition of  $U(\mathfrak{g})$ . Let  $p$  be the projection of  $U(\mathfrak{g})$  onto  $U(\mathfrak{h})$  under this direct sum.

For  $x, y \in U(\mathfrak{g})$ , by invariance,  $\langle xv_{\lambda}, yv_{\lambda} \rangle_{\lambda} = \langle y^*xv_{\lambda}, v_{\lambda} \rangle_{\lambda}$ . Since  $\mathfrak{n}$  acts on  $v_{\lambda}$  by zero, therefore  $\langle U(\mathfrak{g})\mathfrak{n}v_{\lambda}, v_{\lambda} \rangle_{\lambda} = 0$ . As any element of  $U(\mathfrak{g})v_{\lambda}$  is a sum of vectors of weight no more than  $\lambda$ , it follows that any element of  $\mathfrak{n}^{op}U(\mathfrak{g})v_{\lambda}$  is a sum of vectors of weight strictly less than  $\lambda$ . By invariance,  $\langle \mathfrak{n}^{op}U(\mathfrak{g})v_{\lambda}, v_{\lambda} \rangle_{\lambda} = 0$ . We conclude that

$$\langle xv_{\lambda}, yv_{\lambda} \rangle_{\lambda} = \langle p(y^*x)v_{\lambda}, v_{\lambda} \rangle_{\lambda} = \lambda(p(y^*x)) \langle v_{\lambda}, v_{\lambda} \rangle_{\lambda} = \lambda(p(y^*x)).$$

We see from this construction that an invariant Hermitian form on a Verma module is unique up to multiplication by a real scalar.

Let  $v$  and  $w$  be vectors of weight  $\lambda - \mu$  and  $\lambda - \nu$ , respectively. Since

$$\begin{array}{ccc} \langle Hv, w \rangle_\lambda & = & -\langle v, \bar{H}w \rangle_\lambda \\ \parallel & & \parallel \\ (\lambda - \mu)(H) \langle v, w \rangle_\lambda & = & -(\bar{\lambda} - \bar{\nu})(H) \langle v, w \rangle_\lambda = (\lambda + \bar{\nu})(H) \langle v, w \rangle_\lambda, \end{array}$$

we conclude that  $\langle v, w \rangle_\lambda = 0$  if  $\mu \neq -\bar{\nu} = \theta\nu$ . The Shapovalov form pairs the  $\lambda - \mu$  weight space with the  $\lambda - \theta\mu$  weight space. Since the dimension of each weight space of  $M(\lambda)$  is finite, therefore by restricting our attention to each weight space and the weight space to which it is paired individually, we may discuss the signature and the determinant of the Shapovalov form. For the purpose of such a discussion, we study the classical Shapovalov form.

There is a unique involutive automorphism  $\sigma$  of  $\mathfrak{g}$  such that

$$\sigma(X_i) = Y_i, \quad \sigma(Y_i) = X_i, \quad \sigma(H_i) = H_i$$

where the  $X_i, Y_i, H_i$  are the canonical generators of  $\mathfrak{g}$ . It induces an involutive automorphism of  $U(\mathfrak{g})$ , which we will also denote by  $\sigma$ . We know that

$$p(\sigma(x)) = p(x) \quad \forall x \in U(\mathfrak{g})$$

(see [6]). The classical Shapovalov form, which we denote by  $(\cdot, \cdot)_S$ , is defined by

$$(xv_\lambda, yv_\lambda)_S = \lambda(p(\sigma(x)y)) \quad \forall x, y \in U(\mathfrak{g}).$$

It is symmetric, bilinear, and  $(M(\lambda)_{\lambda-\mu}, M(\lambda)_{\lambda-\nu})_S = 0$  if  $\mu \neq \nu$ .

A theorem of Shapovalov states that the determinant of the classical Shapovalov form on the  $\lambda - \mu$  weight space is

$$\prod_{\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{h})} \prod_{n=1}^{\infty} ((\lambda + \rho, \alpha^\vee) - n)^{P(\mu - n\alpha)}$$

up to multiplication by a scalar. Here,  $P$  denotes Kostant's partition function.

Comparing the formulas

$$\langle xv_\lambda, yv_\lambda \rangle_\lambda = \lambda(p(y^*x)) \quad \text{and} \quad (xv_\lambda, yv_\lambda)_S = \lambda(p(\sigma(x)y)) = \lambda(p(\sigma(y)x)),$$

we see that when  $\mu$  is imaginary, the determinant of a matrix representing  $\langle \cdot, \cdot \rangle_\lambda$  on the  $\lambda - \mu$  weight space differs from the classical formula above by the determinant of a change of basis matrix. When  $\mu$  is complex so that the  $\lambda - \mu$  and  $\lambda - \theta\mu$  weight spaces are paired, we see that the form  $\langle \cdot, \cdot \rangle_\lambda$  on  $M(\lambda)_{\lambda-\mu} + M(\lambda)_{\lambda-\theta\mu}$  can be represented by a matrix of the form

$$\begin{matrix} \lambda - \mu \\ \lambda - \theta\mu \end{matrix} \begin{pmatrix} 0 & A \\ \bar{A}^t & 0 \end{pmatrix}$$

where  $A$  and  $\bar{A}^t$  differ from matrices representing the classical Shapovalov form on the  $\lambda - \theta\mu$  and  $\lambda - \mu$  weight spaces, respectively, by multiplication by change of basis matrices. Thus the determinant of this matrix, up to a multiplication by a scalar, is

$$\prod_{\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{h})} \prod_{n=1}^{\infty} ((\lambda + \rho, \alpha^\vee) - n)^{P(\mu - n\alpha)} ((\lambda + \rho, \alpha^\vee) - n)^{P(\theta\mu - n\alpha)}.$$

Unfortunately, when the subspace under consideration has dimension greater than one, a formula for the determinant is insufficient for the purposes of computing the signature.

The radical of the Shapovalov form is the unique maximal submodule of  $M(\lambda)$ , hence the form is non-degenerate precisely for the irreducible Verma modules. The Shapovalov determinant formula indicates precisely where the Shapovalov form is degenerate, and consequently where  $M(\lambda)$  is reducible: on the affine hyperplanes  $H_{\alpha, n} := \{\lambda + \rho \mid (\lambda + \rho, \alpha^\vee) = n\}$  where  $\alpha$  is a positive root and  $n$  is a positive integer. We conclude that in any connected set of purely imaginary  $\lambda$  avoiding these reducibility hyperplanes, as the Shapovalov form never becomes degenerate, the signature corresponding to some fixed  $\mu$  remains constant.

The largest of such regions, which we name the **Wallach region**, is the intersection of the negative open half spaces

$$\left( \bigcap_{\alpha \in \Pi} H_{\alpha, 1}^- \right) \cap H_{\tilde{\alpha}, 1}^-$$

with  $i\mathfrak{h}_0^*$ , where  $\tilde{\alpha}^\vee$  is the highest coroot,  $\Pi$  the set of simple roots corresponding to our choice of  $\Delta^+$ , and  $H_{\beta, n}^- = \{\lambda + \rho \mid (\lambda + \rho, \beta^\vee) < n\}$ .

In [13], Wallach shows for fixed  $\mu$  imaginary that the diagonal entries in a matrix associated to the Shapovalov form  $\langle \cdot, \cdot \rangle_{\lambda+t\xi}$  and the  $\lambda + t\xi - \mu$  weight space have higher degree in  $t$  than the

off-diagonal entries. Thus, choosing  $\lambda$  and  $\xi$  appropriately so that  $\lambda + t\xi$  lies in the Wallach region for all  $t \geq 0$ , an asymptotic argument which examines the signs of the diagonal entries for large  $t$  yields a formula for the signature of the Shapovalov form within the entire Wallach region.

**Definition 2.9.** Denote the signature of the Shapovalov form on the  $\lambda - \mu$  and  $\lambda - (-\bar{\mu})$  weight space(s) of  $M(\lambda)$  by  $(p(\{\mu, -\bar{\mu}\}), q(\{\mu, -\bar{\mu}\}))$ . The **signature character** of  $\langle \cdot, \cdot \rangle_\lambda$  is

$$ch_s M(\lambda) = \sum_{\{\mu, -\bar{\mu}\} \subset \Lambda_r^+} (p(\{\mu, -\bar{\mu}\}) - q(\{\mu, -\bar{\mu}\})) e^{\lambda - \frac{\mu - \bar{\mu}}{2}}$$

where  $\Lambda_r^+$  denotes the positive root lattice.

Here we make the observation that if  $\mu \in \Lambda_r^+$  is complex, then the Shapovalov form pairs the two distinct weight spaces  $M(\lambda)_{\lambda - \mu}$  and  $M(\lambda)_{\lambda - (-\bar{\mu})}$  so that a matrix representing the Shapovalov form on these two weight spaces is of the form

$$\left( \begin{array}{c|c} 0 & A \\ \hline \bar{A}^t & 0 \end{array} \right).$$

The matrix is Hermitian, and so it is diagonalizable with real eigenvalues. Suppose  $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$  is an eigenvector of the matrix of eigenvalue  $r$ . Now

$$\left( \begin{array}{c|c} 0 & A \\ \hline \bar{A}^t & 0 \end{array} \right) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} Av_2 \\ \bar{A}^t v_1 \end{pmatrix} = r \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

and so

$$\left( \begin{array}{c|c} 0 & A \\ \hline \bar{A}^t & 0 \end{array} \right) \begin{pmatrix} v_1 \\ -v_2 \end{pmatrix} = \begin{pmatrix} -Av_2 \\ \bar{A}^t v_1 \end{pmatrix} = -r \begin{pmatrix} v_1 \\ -v_2 \end{pmatrix},$$

giving us an eigenvector of eigenvalue  $-r$ . We conclude if  $\mu$  is complex, then  $p(\{\mu, -\bar{\mu}\})$  and  $q(\{\mu, -\bar{\mu}\})$  are equal. Thus we may write the signature character as

$$ch_s M(\lambda) = \sum_{\substack{\mu \in \Lambda_r^+ \\ \mu \text{ imaginary}}} (p(\mu) - q(\mu)) e^{\lambda - \mu}.$$

**Theorem 2.10.** (Wallach,[13]) *The signature character of  $M(\lambda)$  for  $\lambda + \rho$  in the Wallach region is*

$$ch_s M(\lambda) = \frac{e^\lambda}{\prod_{\alpha \in \Delta^+(\mathfrak{p}, \mathfrak{t})} (1 - e^{-\alpha}) \prod_{\alpha \in \Delta^+(\mathfrak{k}, \mathfrak{t})} (1 + e^{-\alpha})}.$$

This is a rewording of a special case of Lemma 2.3 of [13]. Here, in translating from the language of section 2 of [13] to our language, we choose  $H$  to correspond to  $ir_{\mathfrak{k}}$ . Then  $\mathfrak{q} = \mathfrak{b}$ ,  $\mathfrak{l} = \mathfrak{h}$ ,  $\mathfrak{u} = \mathfrak{n}$ ,  $\mathfrak{u}_n = \bigoplus_{\alpha \in \Delta^+(\mathfrak{p}, \mathfrak{t})} \mathfrak{g}_\alpha$ ,  $\mathfrak{u}_k = \bigoplus_{\alpha \in \Delta^+(\mathfrak{k}, \mathfrak{t})} \mathfrak{g}_\alpha$ ,  $\Delta(\mathfrak{u}_n) = \Delta(\mathfrak{p}, \mathfrak{t})$ , and  $\Delta(\mathfrak{u}_k) = \Delta(\mathfrak{k}, \mathfrak{t})$ . The system of positive roots  $\Phi^+$  for  $(\mathfrak{l} \cap \mathfrak{k}, \mathfrak{t})$  is empty, and therefore the Weyl group  $W_{\mathfrak{l} \cap \mathfrak{k}}$  is trivial,  $\rho_{\mathfrak{l} \cap \mathfrak{k}} = 0$ , and  $D_{\mathfrak{l} \cap \mathfrak{k}} = 1$ . We choose  $V$  to be the trivial representation. Therefore  $D_{\mathfrak{l} \cap \mathfrak{k}} ch_s(V) = 1$ .

Here, we make the observation that the formula for the signature character makes sense due to our results concerning pairings of non-imaginary weight spaces and our characterization of the roots corresponding to a maximally compact Cartan subalgebra.

Our goal is to extend Wallach's result (Theorem 2.10) to all irreducible Verma modules which carry an invariant Hermitian form. The strategy is as follows:

Suppose  $\lambda + \rho$  lies in the hyperplane  $H_{\alpha, n}$ , where  $\alpha$  is a positive root and  $n$  is a positive integer, but for all other positive roots  $\beta$ ,  $(\lambda + \rho, \beta^\vee)$  is not an integer. Then for non-zero  $\xi$  and for non-zero  $t$  in a neighbourhood of 0,  $\langle \cdot, \cdot \rangle_{\lambda + t\xi}$  has radical  $\{0\}$ . Since  $\langle \cdot, \cdot \rangle_\lambda$  has radical isomorphic to the irreducible Verma module  $M(\lambda - n\alpha)$ , the signature must change by plus or minus twice the signature of  $\langle \cdot, \cdot \rangle_{\lambda - n\alpha}$  across  $H_{\alpha, n}$ . (This will be discussed more rigorously in Chapter 3.)

Roughly, by taking a suitable path from  $\lambda$  to the Wallach region and keeping track of changes as we cross reducibility hyperplanes, we arrive at an expression for the signature of  $\langle \cdot, \cdot \rangle_\lambda$  in terms of the signature in the Wallach region.

We shall describe this more concretely in Chapter 4.

## Chapter 3

# The Jantzen filtration

Given a finite-dimensional complex vector space  $E$  and an analytic family  $\langle \cdot, \cdot \rangle_t$  of Hermitian forms defined on  $E$  for  $t \in (-\delta, \delta)$  so that  $\langle \cdot, \cdot \rangle_t$  is non-degenerate for  $t \neq 0$  and degenerate for  $t = 0$ , we define the Jantzen filtration of  $E$  as follows:

$$E = E_0 \supset E_1 \supset \cdots \supset E_N = \{0\}$$

where  $e \in E_n$  for  $n \geq 0$  if there exists an analytic function  $f_e : (-\varepsilon, \varepsilon) \rightarrow E$  for some  $\varepsilon > 0$  such that

1.  $f_e(0) = e$
2.  $\langle f_e(t), e' \rangle_t$  vanishes to order at least  $n$  at  $t = 0$  for any  $e' \in E$ .

For  $e, e' \in E_n$ , define

$$\langle e, e' \rangle^n = \lim_{t \rightarrow 0} \frac{1}{t^n} \langle f_e(t), f_{e'}(t) \rangle_t$$

which is independent of choice of  $f_e$  and  $f_{e'}$ . We have the following results (see section 3 of [12]):

**Theorem 3.1.** (*Vogan, [12]*) *The form  $\langle \cdot, \cdot \rangle^n$  on  $E_n$  is Hermitian with radical  $E_{n+1}$ , and therefore it induces a non-degenerate Hermitian form on  $E_n/E_{n+1}$ , which we also denote  $\langle \cdot, \cdot \rangle^n$ . Let  $(p_n, q_n)$  be the signature of  $\langle \cdot, \cdot \rangle^n$ ,  $(p, q)$  be the signature of  $\langle \cdot, \cdot \rangle_t$  for  $t \in (0, \delta)$ , and  $(p', q')$  be the signature*

of  $\langle \cdot, \cdot \rangle_t$  for  $t \in (-\delta, 0)$ . Then

$$\begin{aligned} p &= p' + \sum_{n \text{ odd}} p_n - \sum_{n \text{ odd}} q_n \\ q &= q' + \sum_{n \text{ odd}} q_n - \sum_{n \text{ odd}} p_n \end{aligned}$$

For the remainder of the chapter, let  $\lambda_t : (-\varepsilon, \varepsilon) \rightarrow i\mathfrak{h}_0^*$  be an analytic map satisfying the following conditions:

1. For some positive root  $\alpha$  and positive integer  $n$ ,  $\lambda_0 \in H_{\alpha, n}$ .
2.  $\lambda_0 \notin H_{\beta, m}$  for  $\beta \neq \alpha, \theta\alpha, \alpha + \theta\alpha$  and  $m$  an integer.
3. For  $t \neq 0$ ,  $\lambda_t$  is imaginary (so the Shapovalov form exists) but does not lie in any reducibility hyperplanes.

We may view  $M(\lambda_t)$  as realized on a fixed vector space  $V$  for every  $t$  in  $(-\varepsilon, \varepsilon)$  via  $M(\lambda_t) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_{\lambda_t} = U(\mathfrak{n}^{op}) \otimes \mathbb{C}_{\lambda_t}$ . From now on, we will identify  $V$  with  $U(\mathfrak{n}^{op})$  and the  $-\mu$  weight space of  $U(\mathfrak{n}^{op})$  with the  $\lambda_t - \mu$  weight space of  $M(\lambda_t)$  without further comment. Since  $\langle xv_{\lambda_t}, yv_{\lambda_t} \rangle_{\lambda_t} = \lambda_t(p(y^*x))$  for  $x, y \in U(\mathfrak{g})$ , therefore  $\langle \cdot, \cdot \rangle_{\lambda_t}$  is an analytic family of Hermitian forms on  $V$ . The Jantzen filtration of  $V$  is

$$V = V_0 \supset V_1 \supset \cdots \supset V_N = \{0\}$$

where  $V_j$  is defined as  $E_j$  was, with the additional stipulation that  $f_e$  take values in a fixed finite-dimensional subspace of  $V$ . As before, we define a Hermitian form  $\langle \cdot, \cdot \rangle^j$  on  $V_j$  with radical  $V_{j+1}$ . We remark that the chain of subspaces is indeed finite as each  $V_j$  is invariant under  $\mathfrak{g}$  and  $M(\lambda_0)$  has finite length.

As we have an  $\mathfrak{h}$ -invariant orthogonal decomposition of  $V$  into finite dimensional subspaces with respect to the Shapovalov form, we may view  $\langle \cdot, \cdot \rangle_{\lambda_t}$  as a collection of analytic families of Hermitian forms on each finite dimensional weight space (or pair of weight spaces) of  $V$ . From orthogonality, we may further conclude that for  $e \in M(\lambda_t)_{\lambda_t - \mu}$ , we may take  $f_e$  to have values in  $M(\lambda_t)_{\lambda_t - \mu}$ . Therefore the Jantzen filtration of  $V$  gives us Jantzen filtrations of each finite dimensional subspace in our orthogonal decomposition of  $V$ , and Theorem 3.1 holds for each of these subspaces. For  $\mu$



imaginary, let  $(p(\mu), q(\mu))$  be the signature of  $\langle \cdot, \cdot \rangle_{\lambda_t}$  on  $M(\lambda_t)_{\lambda_t - \mu}$  for  $t \in (0, \varepsilon)$  and  $(p'(\mu), q'(\mu))$  be the signature for  $t \in (-\varepsilon, 0)$ . Let  $(p_j(\mu), q_j(\mu))$  be the signature of  $\langle \cdot, \cdot \rangle^j$  on the  $-\mu$  weight space of  $V_j/V_{j+1}$ . Then

$$\begin{aligned} p &= p' + \sum_{j \text{ odd}} p_j - \sum_{j \text{ odd}} q_j \\ q &= q' + \sum_{j \text{ odd}} q_j - \sum_{j \text{ odd}} p_j \end{aligned}$$

as before.

In determining the Jantzen filtration of  $V$  corresponding to  $\langle \cdot, \cdot \rangle_{\lambda_t}$ ,  $\mathfrak{g}$ -invariance of the different levels of the filtration establish strong restrictions on the possible values of the  $V_j$ . We have two cases:

**Case 1:**  $\alpha$  is imaginary and  $H_{\alpha, n}$  is the only reducibility hyperplane containing  $\lambda_0$ .

By our choice of  $\lambda_0$ ,  $M(\lambda_0)$  has only one non-trivial submodule:  $M(\lambda_0 - n\alpha)$ . Its multiplicity must be one as  $M(\lambda - n\alpha)$  is a free  $U(\mathfrak{n}^{op})$ -module by choice of  $\lambda$  (see Theorem 7.6.6 of [1]). Therefore our Jantzen filtration must be

$$M(\lambda_0) \supset M(\lambda_0 - n\alpha) \supset \cdots \supset M(\lambda_0 - n\alpha) = V_N \supset \{0\}.$$

According to the Shapovalov determinant formula, up to multiplication by a scalar, the determinant of the form  $\langle \cdot, \cdot \rangle_{\lambda_t}$  on the  $\lambda_t - n\alpha$  weight space is

$$\prod_{m=1}^{\infty} \prod_{\beta \in \Delta^+(\mathfrak{g}, \mathfrak{h})} ((\lambda_t + \rho, \beta^\vee) - m)^{P(n\alpha - m\beta)}.$$

The only factor which is zero when  $t = 0$  is the factor corresponding to  $\beta = \alpha$  and  $m = n$ . Since  $P(0) = 1$ , as we go from  $t > 0$  to  $t < 0$ , the determinant changes sign. Therefore  $N$  must be odd and  $(p_N, q_N)$  or  $(q_N, p_N)$  must be the signature of the Shapovalov form on  $M(\lambda_0 - n\alpha)$ . Thus:

**Proposition 3.2.** *In the setup of this chapter, suppose  $\alpha$  is imaginary. If  $t_1 \in (0, \varepsilon)$  and*

$t_2 \in (-\varepsilon, 0)$ , then

$$ch_s M(\lambda_{t_1}) = e^{\lambda_{t_1} - \lambda_{t_2}} \cdot ch_s M(\lambda_{t_2}) \pm 2ch_s M(\lambda_{t_1} - n\alpha).$$

**Case 2:**  $\alpha$  is complex (so  $\lambda_0$  is contained in both  $H_{\alpha, n}$ ,  $H_{\theta\alpha, n}$ , and also in  $H_{\alpha + \theta\alpha, 2n}$  if  $\alpha + \theta\alpha$  is a root).

We know that  $M(\lambda_0 - n\alpha)$  is a submodule of  $M(\lambda_0)$  when  $(\lambda_0 + \rho, \alpha^\vee) = n$ . As  $\lambda_0$  and  $\rho$  are imaginary, therefore  $(\lambda_0 + \rho, -\bar{\alpha}^\vee) = -(\overline{\lambda_0 + \rho}, \alpha^\vee) = \overline{(\lambda_0 + \rho, \alpha^\vee)} = \bar{n} = n$ . Hence  $M(\lambda_0 - n(-\bar{\alpha}))$  must also be a submodule of  $M(\lambda_0)$ .

Key to describing the Jantzen filtration in this case is the usage of results of Bernstein, Gelfand, and Gelfand, described in [1]. Let  $J(\lambda)$  denote the unique largest submodule of  $M(\lambda)$  and  $L(\lambda) = M(\lambda)/J(\lambda)$  the corresponding simple quotient.

**Proposition 3.3.** (*Proposition 7.6.1, [1]*) *The Verma module  $M(\lambda)$  has a Jordan-Hölder series and every simple subquotient of  $M(\lambda)$  is isomorphic to  $L(\mu)$  for some  $\mu$  belonging to  $W \cdot (\lambda + \rho) \cap (\lambda + \rho - \Lambda_r^+) - \rho$ .*

Beware that the notation in [1] includes a shift by  $\rho$ .

**Theorem 3.4.** (*Theorem 7.6.6, [1]*) *For  $\mu, \lambda \in \mathfrak{h}^*$ ,*

$$\dim \text{Hom}(M(\mu), M(\lambda)) \leq 1.$$

**Theorem 3.5.** (*Bernstein-Gelfand-Gelfand, Theorem 7.6.23 of [1]*) *For  $\lambda, \mu \in \mathfrak{h}^*$ ,*

$$\begin{aligned} M(\mu) \subset M(\lambda) &\iff \exists \alpha_1, \dots, \alpha_m \in \Delta^+(\mathfrak{g}, \mathfrak{h}) \text{ such that} \\ \lambda + \rho &\geq s_{\alpha_1}(\lambda + \rho) \geq \dots \geq s_{\alpha_m} \dots s_{\alpha_1}(\lambda + \rho) = \mu + \rho. \end{aligned}$$

The above conditions are equivalent to

$$\mu + \rho \in W(\lambda + \rho) \text{ and } \mu \leq \lambda$$

in the case where  $\mathfrak{g}$  is type  $A_2$  (see remark 7.8.10, [1]).

**If  $\alpha$  and  $-\bar{\alpha} = \theta\alpha$  are orthogonal:** We have  $(\lambda_0 - n\alpha + \rho, (\theta\alpha)^\vee) = (\lambda_0 + \rho, (\theta\alpha)^\vee) = n$ .

By symmetry and our discussion above, we have the following containment of Verma modules:

$$\begin{array}{ccc}
 & M(\lambda_0) & \\
 \swarrow & & \searrow \\
 M(\lambda_0 - n\alpha) & & M(\lambda_0 - n\theta\alpha) \\
 \searrow & & \swarrow \\
 & M(\lambda_0 - n(\alpha + \theta\alpha)) &
 \end{array}$$

The radical of the Shapovalov form on  $M(\lambda_0)$  is the unique largest proper submodule of  $M(\lambda_0)$ , and so the radical of  $\langle \cdot, \cdot \rangle_0$  is  $\{M(\lambda_0 - n\alpha), M(\lambda_0 - n\theta\alpha)\}$ , the submodule generated by  $M(\lambda_0 - n\alpha)$  and  $M(\lambda_0 - n\theta\alpha)$ . It is also equal to  $V_1$ , the first level of our Jantzen filtration. We have an invariant Hermitian form  $\langle \cdot, \cdot \rangle^1$  on  $V_1$ .

Let  $v_{\lambda_0 - n\alpha}$  and  $v_{\lambda_0 - n\theta\alpha}$  be generators of  $M(\lambda_0 - n\alpha)$  and  $M(\lambda_0 - n\theta\alpha)$ , respectively. Recall that if  $v$  and  $w$  are vectors of weight  $\mu$  and  $\nu$ , respectively, then  $\langle v, w \rangle^1 = 0$  if  $\nu \neq -\bar{\mu}$  due to invariance. Observe that for any monomials  $x, y \in U(\mathfrak{n}^{op})$  of weights  $\mu$  and  $-\bar{\mu} - n\bar{\alpha} - n\alpha$  respectively,

$$\langle xv_{\lambda_0 - n\alpha}, yv_{\lambda_0 - n\alpha} \rangle^1 = \langle y^* xv_{\lambda_0 - n\alpha}, v_{\lambda_0 - n\alpha} \rangle^1 = 0$$

as  $y^*x$  has weight  $n(\alpha - \bar{\alpha}) = n(\alpha + \theta\alpha) > 0$  and  $v_{\lambda_0 - n\alpha}$  is singular. Hence

$$\langle M(\lambda_0 - n\alpha), M(\lambda_0 - n\alpha) \rangle^1 = 0.$$

Similarly,

$$\langle M(\lambda_0 - n\theta\alpha), M(\lambda_0 - n\theta\alpha) \rangle^1 = 0,$$

whence  $M(\lambda_0 - n(\alpha + \theta\alpha)) \subset M(\lambda_0 - n\alpha) \cap M(\lambda_0 - n\theta\alpha)$  is contained in the radical of  $\langle \cdot, \cdot \rangle^1$ . The Jantzen filtration must be

$$\begin{aligned}
 M(\lambda_0) &\supset \{M(\lambda_0 - n\alpha), M(\lambda_0 - n\theta\alpha)\} \supset \cdots \supset \{M(\lambda_0 - n\alpha), M(\lambda_0 - n\theta\alpha)\} = V_M \\
 &\supset M(\lambda_0 - n(\alpha + \theta\alpha)) \supset \cdots \supset M(\lambda_0 - n(\alpha + \theta\alpha)) = V_N \supset \{0\}
 \end{aligned}$$

for some  $M$  and  $N$ .

Whether  $M$  is even or odd, the contribution  $p_M - q_M$  to the signature character from the  $M^{\text{th}}$  level of the filtration is zero as  $M(\lambda_0 - n\alpha) \setminus M(\lambda_0 - n\theta\alpha)$  is paired with  $M(\lambda_0 - n\theta\alpha) \setminus M(\lambda_0 - n\alpha)$ .

Up to multiplication by a scalar, the determinant of a matrix representing  $\langle \cdot, \cdot \rangle_{\lambda_t}$  on the  $\lambda_t - n(\alpha + \theta\alpha)$  weight space of  $M(\lambda_t)$  is

$$\prod_{m=1}^{\infty} \prod_{\beta \in \Delta^+(\mathfrak{g}, \mathfrak{h})} ((\lambda_t + \rho, \beta^\vee) - m)^{P(n(\alpha + \theta\alpha) - m\beta)}.$$

The only factors which are zero when  $t = 0$  are those corresponding to the pairs  $(\alpha, n)$  and  $(\theta\alpha, n)$ . Observe that  $P(n\alpha) = P(n\theta\alpha)$  as  $\theta$  is a bijection from  $\Delta^+(\mathfrak{g}, \mathfrak{h})$  to itself. Combined with the fact that  $(\lambda_t + \rho, \alpha^\vee) = (\lambda_t + \rho, (\theta\alpha)^\vee)$ , we see that the determinant does not change as  $t$  changes from positive to negative. In other words,  $N$  must be even. We have:

**Proposition 3.6.** *In the setup of this chapter, suppose  $\alpha$  is complex and  $\alpha$  and  $\theta\alpha$  are orthogonal. Then for  $t_1 \in (0, \varepsilon)$  and  $t_2 \in (-\varepsilon, 0)$ ,*

$$ch_s M(\lambda_{t_1}) = e^{\lambda_{t_1} - \lambda_{t_2}} \cdot ch_s M(\lambda_{t_2}).$$

**If  $\alpha$  and  $-\bar{\alpha} = \theta\alpha$  are not orthogonal:** Now  $\alpha$  and  $-\bar{\alpha} = \theta\alpha$  have the same length. If  $\alpha$  and  $\theta\alpha$  are not orthogonal, then either  $(\alpha, (\theta\alpha)^\vee) = \pm 1$  or  $(\theta\alpha, \alpha^\vee) = \pm 1$ , whence either  $\alpha + \theta\alpha$  or  $\alpha - \theta\alpha$  is a root. Observe that  $\alpha$  and  $\theta\alpha$  have the same height as  $\theta$  applied to an expression for  $\alpha$  as a sum of indecomposable roots gives an expression for  $\theta\alpha$  as a sum of indecomposable roots (we will see such an argument again in Chapter 6). We conclude that  $\alpha - \theta\alpha$  cannot be a root. Thus  $\alpha + \theta\alpha$  must be a root and  $\alpha$  and  $\theta\alpha$  generate a subroot system of type  $A_2$ . As above,  $(\lambda_0 + \rho, (\theta\alpha)^\vee) = (\lambda_0 + \rho, \alpha^\vee) = n$ , which implies  $(\lambda_0 + \rho, (\alpha + \theta\alpha)^\vee) = (\lambda_0 + \rho, \alpha^\vee) + (\lambda_0 + \rho, (\theta\alpha)^\vee) = 2n$ . It follows that

$M(\lambda_0 - 2n(\alpha + \theta\alpha))$  is a submodule of  $M(\lambda_0)$ . From

$$\begin{aligned} (\lambda_0 - n\alpha + \rho, (\theta\alpha)^\vee) &= 2n & (\lambda_0 - n\alpha - 2n\theta\alpha + \rho, \alpha^\vee) &= 2n \\ (\lambda_0 - n\alpha + \rho, (\alpha + \theta\alpha)^\vee) &= n & (\lambda_0 - n\alpha - 2n\theta\alpha + \rho, (\theta\alpha)^\vee) &= 0 \end{aligned}$$

and from symmetry between  $\alpha$  and  $\theta\alpha$ , we observe the following containment of Verma modules:

$$\begin{array}{ccccc} & & M(\lambda_0) & & \\ & \swarrow & & \searrow & \\ M(\lambda_0 - n\alpha) & & & & M(\lambda_0 - n\theta\alpha) \\ & \searrow & & \swarrow & \\ & M(\lambda_0 - n\alpha) \cap M(\lambda_0 - n\theta\alpha) & & & \\ & \swarrow & & \searrow & \\ M(\lambda_0 - n\alpha - 2n\theta\alpha) & & & & M(\lambda_0 - 2n\alpha - n\theta\alpha) \\ & \searrow & & \swarrow & \\ & M(\lambda_0 - 2n(\alpha + \theta\alpha)) & & & \end{array}$$

By choice of  $\lambda_t$ , our remarks at the beginning of Case 2, and arguments similar to those of the above subcase, our Jantzen filtration must be

$$\begin{aligned} M(\lambda_0) &\supset \{M(\lambda_0 - n\alpha), M(\lambda_0 - n\theta\alpha)\} \supset \cdots \supset \{M(\lambda_0 - n\alpha), M(\lambda_0 - n\theta\alpha)\} = V_{N_1} \\ &\supset M(\lambda_0 - n\alpha) \cap M(\lambda_0 - n\theta\alpha) = \{M(\lambda_0 - n\alpha - 2n\theta\alpha), M(\lambda_0 - 2n\alpha - n\theta\alpha)\} \\ &\supset \cdots \supset \{M(\lambda_0 - n\alpha - 2n\theta\alpha), M(\lambda_0 - 2n\alpha - n\theta\alpha)\} = V_{N_2} \\ &\supset M(\lambda_0 - 2n(\alpha + \theta\alpha)) \supset \cdots \supset M(\lambda_0 - 2n(\alpha + \theta\alpha)) = V_{N_3} \supset \{0\} \end{aligned}$$

As in the previous subcase, the  $N_1^{\text{th}}$  and  $N_2^{\text{th}}$  levels of the Jantzen filtration give no contribution to the change in signature character across  $H_{\alpha,n}$ .

We study the determinant of  $\langle \cdot, \cdot \rangle_{\lambda_t}$  on the  $\lambda_t - 2n(\alpha + \theta\alpha)$  weight space of  $M(\lambda_t)$ :

$$\prod_{m=1}^{\infty} \prod_{\beta \in \Delta^+(\mathfrak{g}, \mathfrak{h})} ((\lambda_t + \rho, \beta^\vee) - m)^{P(2n(\alpha + \theta\alpha) - m\beta)}.$$

The pairs  $(\beta, m)$  for which the corresponding factor is zero at  $t = 0$  are  $(\alpha, n)$ ,  $(\theta\alpha, n)$ , and  $(\alpha + \theta\alpha, 2n)$ . Again, as  $(\lambda_t + \rho, \alpha) = (\lambda_t + \rho, \theta\alpha)$ ,  $P(n\alpha + 2n\theta\alpha) = P(2n\alpha + n\theta\alpha)$ , and  $P(0) = 1$ ,  $N_3$  must be odd. We have the following:

**Proposition 3.7.** *In the setup of this chapter, suppose  $\alpha$  is complex and  $\alpha$  and  $\theta\alpha$  are not orthogonal so that  $\alpha + \theta\alpha$  is an imaginary root. Then for  $t_1 \in (0, \varepsilon)$  and  $t_2 \in (-\varepsilon, 0)$ ,*

$$ch_s M(\lambda_{t_1}) = e^{\lambda_{t_1} - \lambda_{t_2}} \cdot ch_s M(\lambda_{t_2}) \pm 2ch_s M(\lambda_{t_1} - n(\alpha + \theta\alpha)).$$

**Remark 3.8.** *This agrees with Proposition 3.2.*

## Chapter 4

# A preliminary formula for the signature character

In this and the subsequent chapter, we will assume that  $\mathfrak{h}$  is a compact Cartan subalgebra—that is,  $\mathfrak{h} = \mathfrak{t}$  and  $\mathfrak{a} = 0$ . Then *all* roots are imaginary.

**Definition 4.1.** *According to Theorem 2.10, there are constants  $c_\mu$  for  $\mu \in \Lambda_r^+$  so that*

$$R(\lambda) := \sum_{\mu \in \Lambda_r^+} c_\mu e^{\lambda - \mu}$$

*is the signature character of the Shapovalov form  $\langle \cdot, \cdot \rangle_\lambda$  when  $\lambda + \rho$  lies in the Wallach region.*

Consider  $A_0 = \{\lambda + \rho \mid (\lambda + \rho, \alpha^\vee) < 0 \ \forall \alpha \in \Pi, \ (\lambda + \rho, \tilde{\alpha}^\vee) > -1\}$ , which we call the **fundamental alcove**. Reflections through the walls of the fundamental alcove generate the affine Weyl group,  $W_a$ . The action of the affine Weyl group defines alcoves which have walls of the form  $H_{\alpha, n}$ . (See [4].) Note that the signature of the Shapovalov form does not change within each of these alcoves.

**Definition 4.2.** *For an alcove  $A$ , there are constants  $c_\mu^A$  for  $\mu \in \Lambda_r^+$  such that*

$$R^A(\lambda) := \sum_{\mu \in \Lambda_r^+} c_\mu^A e^{\lambda - \mu}$$

*is the signature character of the Shapovalov form  $\langle \cdot, \cdot \rangle_\lambda$  when  $\lambda + \rho$  lies in the alcove  $A$ .*

**Lemma 4.3.** *If  $wA_0$  and  $w'A_0$  are adjacent alcoves separated by the hyperplane  $H_{\alpha,n}$ , then*

$$R^{wA_0}(\lambda) = R^{w'A_0}(\lambda) + 2\varepsilon(wA_0, w'A_0)R^{wA_0-n\alpha}(\lambda - n\alpha) \quad (4.1)$$

where  $\varepsilon(wA_0, w'A_0)$  is zero if  $H_{\alpha,n}$  is not a reducibility hyperplane and plus or minus one otherwise.

*Proof.* This is just Proposition 3.2. □

**Remark 4.4.** *Calculating  $\varepsilon$  is difficult and will be the subject of the following chapter.*

**Remark 4.5.** *Observe that  $\varepsilon(wA_0, w'A_0) = -\varepsilon(w'A_0, wA_0)$ .*

Recall that the reflections through the walls of  $A_0$  generate  $W_a$ . These reflections are denoted by  $s_{\alpha,0}$  for each simple root  $\alpha$  and  $s_{\tilde{\alpha},-1}$ . If we omit  $s_{\tilde{\alpha},-1}$ , we generate the Weyl group  $W$  as a subgroup of  $W_a$ . These generators are compatible with reflection through the walls of the fundamental Weyl chamber  $\mathfrak{C}_0$ , which we choose to be the Weyl chamber which contains  $A_0$ :  $\mathfrak{C}_0 = \bigcap_{\alpha \in \Pi} H_{\alpha,0}^-$ . Observe that for each  $s \in W$ ,  $sA_0$  lies in the Wallach region so that  $R^{sA_0} = R$ .

We will define two maps  $\bar{\cdot}$  and  $\tilde{\cdot}$  from the affine Weyl group to the Weyl group as follows: If  $w = ts$  where  $s$  is an element of the Weyl group and  $t$  is translation by an element of the root lattice, then  $\bar{w} = s$ . We let  $\tilde{w}$  be such that  $wA_0$  lies in the Weyl chamber  $\tilde{w}\mathfrak{C}_0$ . Observe that  $\bar{\cdot}$  is a group homomorphism while  $\tilde{\cdot}$  is not. Furthermore,  $\overline{s_{\alpha,n}} = s_{\alpha}$ . Observe that we can rewrite (4.1) as

$$\begin{aligned} R^{wA_0}(\lambda) &= R^{w'A_0}(\lambda) + 2\varepsilon(wA_0, w'A_0)R^{s_{\alpha,0}s_{\alpha,n}wA_0}(s_{\alpha,0}s_{\alpha,n}\lambda) \\ &= R^{w'A_0}(\lambda) + 2\varepsilon(wA_0, w'A_0)R^{\overline{s_{\alpha,n}w'A_0}}(\overline{s_{\alpha,n}w'A_0}\lambda). \end{aligned} \quad (4.2)$$

For  $w$  in the affine Weyl group, let  $wA_0 = C_0 \xrightarrow{r_1} C_1 \xrightarrow{r_2} \dots \xrightarrow{r_\ell} C_\ell = \tilde{w}A_0$  be a (not necessarily reduced) path from  $wA_0$  to  $\tilde{w}A_0$ . Applying (4.2)  $\ell$  times, we obtain

$$\begin{aligned} R^{wA_0}(\lambda) &= R^{\tilde{w}A_0}(\lambda) + \sum_{j=1}^{\ell} \varepsilon(C_{j-1}, C_j) 2R^{\overline{r_j}C_j}(\overline{r_j}r_j\lambda) \\ &= R(\lambda) + 2 \sum_{j=1}^{\ell} \varepsilon(C_{j-1}, C_j) R^{\overline{r_j}C_j}(\overline{r_j}r_j\lambda). \end{aligned}$$

Observe that a path from  $\overline{r_j}C_j$  to  $\overline{r_j}C_\ell$  is  $\overline{r_j}C_j \xrightarrow{\overline{r_j}r_{j+1}\overline{r_j}} \overline{r_j}C_{j+1} \xrightarrow{\overline{r_j}r_{j+2}\overline{r_j}} \dots \xrightarrow{\overline{r_j}r_\ell\overline{r_j}} \overline{r_j}C_\ell$ . Applying



induction on path length, we arrive at the following:

**Theorem 4.6.** Recall  $R : \lambda \mapsto \sum_{\mu \in \Lambda_r^+} c_\mu e^{\lambda - \mu}$  which was defined to agree with the signature character of the Shapovalov form in the Wallach region and  $R^{wA_0} : \lambda \mapsto \sum_{\mu \in \Lambda_r^+} c_\mu^{wA_0} e^{\lambda - \mu}$  which was defined to agree with the signature character of the Shapovalov form within the alcove  $wA_0$ .  
Let  $wA_0 = C_0 \xrightarrow{r_1} C_1 \xrightarrow{r_2} \dots \xrightarrow{r_\ell} C_\ell = \tilde{w}A_0$  be a (not necessarily reduced) path from  $wA_0$  to  $\tilde{w}A_0$ .  
Then

$$\begin{aligned} R^{wA_0}(\lambda) &= \sum_{I=\{i_1 < \dots < i_k\} \subset \{1, \dots, \ell\}} \varepsilon(I) 2^{|I|} R^{\overline{r_{i_1}} \dots \overline{r_{i_k}} \tilde{w}A_0} (\overline{r_{i_1}} r_{i_2} \dots \overline{r_{i_k}} r_{i_k} r_{i_{k-1}} \dots r_{i_1} \lambda) \\ &= \sum_{I=\{i_1 < \dots < i_k\} \subset \{1, \dots, \ell\}} \varepsilon(I) 2^{|I|} R(\overline{r_{i_1}} r_{i_2} \dots \overline{r_{i_k}} r_{i_k} r_{i_{k-1}} \dots r_{i_1} \lambda) \end{aligned}$$

where  $\varepsilon(\emptyset) = 1$  and  $\varepsilon(I) = \varepsilon(C_{i_1-1}, C_{i_1}) \varepsilon(\overline{r_{i_1}} C_{i_2-1}, \overline{r_{i_1}} C_{i_2}) \dots \varepsilon(\overline{r_{i_1}} \dots \overline{r_{i_{k-1}}} C_{i_k-1}, \overline{r_{i_1}} \dots \overline{r_{i_{k-1}}} C_{i_k})$ .

We will determine  $\varepsilon(C, C')$  using the principle that in a closed loop, the changes introduced by crossing reducibility hyperplanes must sum to zero. We know  $R$  by Wallach's work (Theorem 2.10). Theorem 4.6 will therefore give an explicit formula for the signature character of the Shapovalov form on  $M(\lambda)$ , where  $\lambda + \rho$  lies in  $wA_0$ . This solves the problem of calculating the signature for all irreducible Verma modules which admit an invariant Hermitian form.



## Chapter 5

# Calculating $\varepsilon$

The strategy for computing  $\varepsilon$  is as follows:

- We show that for a fixed hyperplane  $H_{\alpha,n}$ , the value of  $\varepsilon$  for crossing from  $H_{\alpha,n}^+$  to  $H_{\alpha,n}^-$  depends only on the Weyl chamber to which the point of crossing belongs.
- We consider rank 2 root systems of types  $A_2$  and  $B_2$ , generated by simple roots  $\alpha_1$  and  $\alpha_2$ , and calculate the values of  $\varepsilon$  by calculating changes that occur at the Weyl chamber walls. It is trivial to show by considering appropriate weight vectors in the Verma module that  $\varepsilon$  for a hyperplane corresponding to a simple root is constant and does not depend on Weyl chambers in any way. However, we prove this in a manner that does not depend on simplicity of the  $\alpha_i$ . We assume that our root system is not of type  $G_2$  in the following, as  $G_2$  is not a proper subroot system of any simple root system.
- For an arbitrary positive root  $\gamma$  in a generic irreducible root system which is not type  $G_2$ , we develop a formula for  $\varepsilon$  inductively by replacing the  $\alpha_i$  from the previous step with appropriate roots. Key in the induction is the independence of our rank 2 arguments from the simplicity of the  $\alpha_i$ .

## 5.1 Dependence on Weyl chambers

We begin by refining Theorem 4.6: if we take an arbitrary  $C_\ell$ , the formula becomes

$$R^{wA_0}(\lambda) = \sum_{I=\{i_1 < \dots < i_k\} \subset \{1, \dots, \ell\}} \varepsilon(I) 2^{|I|} R^{\overline{r_{i_1}} \dots \overline{r_{i_k}} C_\ell} (\overline{r_{i_1}} \overline{r_{i_2}} \dots \overline{r_{i_k}} r_{i_k} r_{i_{k-1}} \dots r_{i_1} \lambda).$$

If we choose in particular  $C_\ell = C_0$ , we have

$$R^{C_0}(\lambda) = \sum_{\substack{I=\{i_1 < \dots < i_k\} \\ \subset \{1, \dots, \ell\}}} \varepsilon(I) 2^{|I|} R^{\overline{r_{i_1}} \dots \overline{r_{i_k}} C_0} (\overline{r_{i_1}} \dots \overline{r_{i_k}} r_{i_k} \dots r_{i_1} \lambda). \quad (5.1.1)$$

**Proposition 5.1.1.** *Suppose  $\alpha$  is a positive root and  $n \in \mathbb{Z}^+$  and suppose  $H_{\alpha,n}$  separates adjacent alcoves  $wA_0$  and  $w'A_0$ , with  $wA_0 \subset H_{\alpha,n}^+$  and  $w'A_0 \subset H_{\alpha,n}^-$ . The value of  $\varepsilon(w, w')$  depends only on  $H_{\alpha,n}$  and on  $\tilde{w}$ .*

*Proof.* We begin by proving the proposition in the special case where  $wA_0 = C_i$  and  $w'A_0 = C_{i+1}$  as described in the following figure:

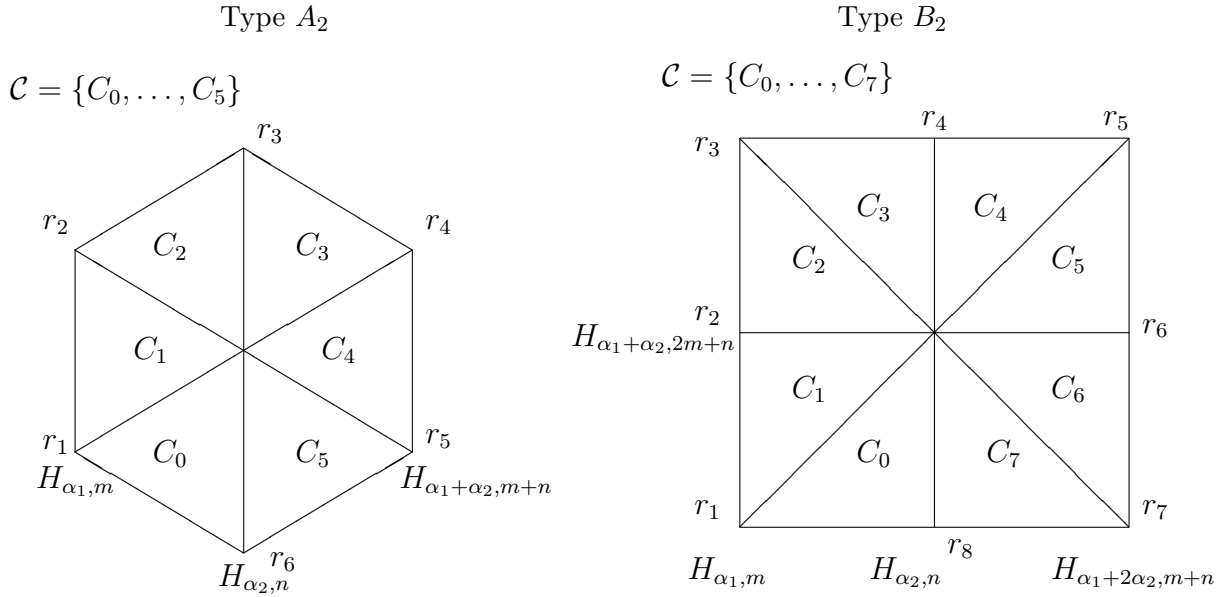


Figure 5-1: Classical rank 2 systems

As we may cover any hyperplane with overlapping translates of  $\mathcal{C}$ , it suffices to show that  $\varepsilon(C_i, C_{i+1}) + \varepsilon(C_{i+\ell/2}, C_{i+1+\ell/2}) = 0$  for  $i = 0, 1, \dots, \ell - 1$  in these rank 2 cases. To show this, we

need the following result:

**Claim 5.1.2.** *Let  $\mathcal{C} = \{C_i\}_{i=0,\dots,\ell-1}$  be a set of alcoves that lie in the interior of some Weyl chamber and suppose the reflections  $\{r_j\}_{j=1,\dots,k}$  preserve  $\mathcal{C}$ . If  $w, v \in W_a$  are generated by the  $r_j$  then*

$$\bar{w}^{-1}w = \bar{v}^{-1}v \iff w = v.$$

*Proof.*  $\Rightarrow$ : By simple transitivity of the action of  $W_a$  on the alcoves,  $\bar{w}^{-1}w = \bar{v}^{-1}v$  if and only if  $\bar{w}^{-1}wC = \bar{v}^{-1}vC$  for any alcove  $C$ . Choose in particular  $C = C_i$ . The alcoves  $\bar{w}^{-1}wC_i$  and  $\bar{v}^{-1}vC_i$  belong to the same Weyl chamber as they are the same alcove. As the  $r_j$ 's preserve  $\mathcal{C}$  which lies in the interior of some Weyl chamber,  $wC_i$  and  $vC_i$  belong to the same Weyl chamber  $s\mathfrak{C}_0$ , say. Therefore the Weyl chamber containing  $\bar{w}^{-1}wC_i = \bar{v}^{-1}vC_i$  may be expressed both as  $\bar{w}^{-1}s\mathfrak{C}_0$  and as  $\bar{v}^{-1}s\mathfrak{C}_0$ . It follows that  $\bar{w}^{-1} = \bar{v}^{-1}$ , whence  $w = v$ . The other direction is trivial.  $\square$

We return to proving  $\varepsilon(C_i, C_{i+}) + \varepsilon(C_{i+\ell/2}, C_{i+1+\ell/2}) = 0$  for  $i = 0, 1, \dots, \ell - 1$  in our rank 2 cases.

For  $I = \{i_1 < \dots < i_k\}$ , we define  $w_I = r_{i_k} r_{i_{k-1}} \dots r_{i_1}$ . We rewrite (5.1.1) as

$$\sum_{\emptyset \neq I \subset \{1, \dots, \ell\}} 2^{|I|} \varepsilon(I) R^{\bar{w}_I^{-1}C_0} (\bar{w}_I^{-1}w_I \lambda) = 0 \quad (5.1.2)$$

Our rank 2 cases satisfy the conditions for Claim 5.1.2. Using Claim 5.1.2 and the partial ordering on  $\Lambda$ , we obtain

$$\sum_{\substack{\emptyset \neq I \subset \{1, \dots, \ell\} \\ \bar{w}_I^{-1}w_I = \mu}} 2^{|I|} \varepsilon(I) = 0 \quad (5.1.3)$$

for every  $\mu \in \Lambda$ .

Suppose  $\mu = m\alpha_1$ . The subsets  $I$  of length less than three for which  $\bar{w}_I^{-1}w_I = m\alpha_1$  are  $I = \{1\}, \{1 + \ell/2\}$ . By considering equation (5.1.3) modulo 8, we obtain

$$\varepsilon(C_0, C_1) + \varepsilon(C_{\ell/2}, C_{\ell/2+1}) = 0,$$

which gives the desired result for  $H_{\alpha_1, m}$ . The same proof can be used for the other hyperplanes. (Note that this proof works for type  $G_2$  also.)

To extend the proof of this proposition to the general case where  $\Delta(\mathfrak{g}, \mathfrak{h})$  is any irreducible root system other than  $G_2$ , we consider an arbitrary positive root  $\alpha$ . There exists some positive  $\beta$  distinct from  $\alpha$  such that  $(\alpha, \beta) \neq 0$ . Then  $\alpha$  and  $\beta$  generate a rank 2 root subsystem of type  $A_2$  or  $B_2$ . Consider two-dimensional affine planes of the form  $P = \text{span}\{\alpha, \beta\} + \mu_0$ . We may choose  $\mu_0$  to lie in the intersection of the hyperplanes  $H_{\alpha, n}$  and  $H_{\beta, m}$ . The intersection of  $H_{\alpha, n}$  and  $H_{\beta, m}$  with  $P$  looks like Figure 5-1, with the possible inclusion of additional affine hyperplanes.

Consider roots  $\delta$  that do not belong to the subsystem generated by  $\alpha$  and  $\beta$ . If  $\delta$  is orthogonal to  $\alpha$  and to  $\beta$ , then  $P \subset H_{\delta, k}$  if  $(\mu_0, \delta^\vee) = k$ , and  $P \cap H_{\delta, k} = \emptyset$  otherwise. We restrict our attention for now to the case where  $P$  has trivial intersection with reducibility hyperplanes corresponding to roots orthogonal to  $\alpha$  and to  $\beta$ . For a root  $\delta$  for which  $(\delta, \alpha) \neq 0$  or  $(\delta, \beta) \neq 0$ ,  $H_{\delta, k}$  intersects  $P$  in a line. Whenever we have an intersection of reducibility hyperplanes in a point  $\mu_0$  in  $P$  that does not lie in any Weyl chamber wall, we may take the alcoves  $C_i$  and the reflections  $r_i$  to correspond to a circular path in  $P$  around  $\mu_0$  of suitably small radius, and we take  $\mathcal{C} \supset \{C_i\}$  to be the set of alcoves containing  $\mu_0$  in their boundaries, so that  $r_i$  preserves  $\mathcal{C}$ . Then, the conditions of Lemma 5.1.1 are satisfied, so we may argue as before and conclude that the signs corresponding to alcoves in the circular path agree with the proposition.

In the following diagrams, solid lines correspond to roots in the subsystem generated by  $\alpha$  and  $\beta$ ; dotted lines correspond to various  $\delta$ .

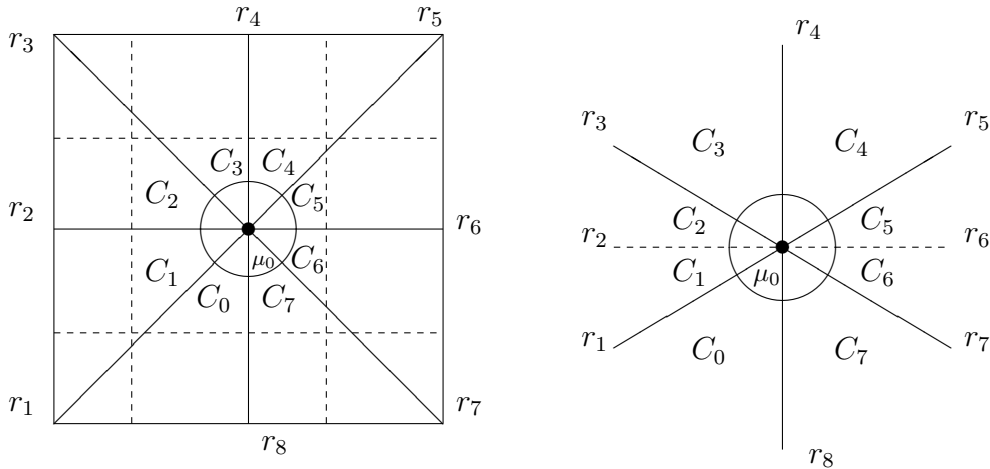
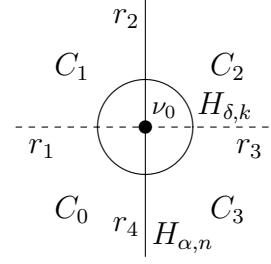


Figure 5-2: Some examples

We partition a given Weyl chamber into regions by hyperplanes  $H_{\delta,k}$  for positive integers  $k$  and positive roots  $\delta$  orthogonal to  $\alpha$  and to  $\beta$ . We conclude from our discussion above that for any pair of adjacent alcoves  $wA_0$  and  $w'A_0$  belonging to a given region, the value of  $\varepsilon(wA_0, w'A_0)$  is the same, whenever the alcoves are separated by  $H_{\alpha,n}$ ,  $wA_0 \subset H_{\alpha,n}^+$ , and  $w'A_0 \subset H_{\alpha,n}^-$ .

To obtain our result for the entire Weyl chamber, consider a reducibility hyperplane  $H_{\delta,k}$  for which  $\delta$  is orthogonal to both  $\alpha$  and  $\beta$ . Take  $\nu_0$  in the intersection of  $H_{\delta,k}$  with  $H_{\alpha,n}$  such that  $\nu_0$  lies in the Weyl chamber under consideration and  $(\nu_0, \gamma^\vee)$  is not an integer for roots  $\gamma$  not equal to plus or minus  $\alpha$  or  $\delta$ . Then, taking a circular path in  $\text{span}\{\alpha, \delta\} + \nu_0$  around  $\nu_0$  of suitably small radius, we may argue as above to conclude that the value for  $\varepsilon$  corresponding to crossing  $H_{\alpha,n}$  in the region bounded by  $H_{\delta,k-1}$  and  $H_{\delta,k}$  is the same as the value for  $\varepsilon$  corresponding to crossing  $H_{\alpha,n}$  in the region bounded by  $H_{\delta,k}$  and  $H_{\delta,k+1}$ .



## 5.2 Calculating $\varepsilon$ for the rank 2 case

**Proposition 5.2.1.** *Using the setup as defined in figure 5-1:*

**Type  $A_2$ :**

Weyl chamber walls in $\mathcal{C}$	Equations
$H_{\alpha_1,0}$	$\varepsilon(C_2, C_3) + \varepsilon(C_5, C_6) = 0$ $\varepsilon(C_1, C_2) + \varepsilon(C_4, C_5) + 2\varepsilon(C_2, C_3)\varepsilon(\overline{r_3}C_4, \overline{r_3}C_5) = 0$
$H_{\alpha_2,0}$	$\varepsilon(C_0, C_1) + \varepsilon(C_3, C_4) = 0$ $\varepsilon(C_1, C_2) + \varepsilon(C_4, C_5) + 2\varepsilon(C_0, C_1)\varepsilon(\overline{r_1}C_1, \overline{r_1}C_2) = 0$
$H_{\alpha_1+\alpha_2,0}$	$\varepsilon(C_0, C_1) + \varepsilon(C_3, C_4) = 0$ $\varepsilon(C_2, C_3) + \varepsilon(C_5, C_6) = 0$

**Type  $B_2$ :**

Weyl chamber walls in $\mathcal{C}$	Equations
$H_{\alpha_1,0}$	$\varepsilon(C_2, C_3) + \varepsilon(C_6, C_7) = 0$ $\varepsilon(C_3, C_4) + \varepsilon(C_7, C_0) = 0$ $\varepsilon(C_1, C_2) + \varepsilon(C_5, C_6) + 2\varepsilon(C_3, C_4)\varepsilon(\overline{r_4}C_6, \overline{r_4}C_7) = 0$
$H_{\alpha_2,0}$	$\varepsilon(C_0, C_1) + \varepsilon(C_4, C_5) = 0$ $\varepsilon(C_1, C_2) + \varepsilon(C_5, C_6) = 0$ $\varepsilon(C_2, C_3) + \varepsilon(C_6, C_7) + 2\varepsilon(C_0, C_1)\varepsilon(\overline{r_1}C_1, \overline{r_1}C_2) = 0$
$H_{\alpha_1+\alpha_2,0}$	$\varepsilon(C_0, C_1) + \varepsilon(C_4, C_5) = 0$ $\varepsilon(C_3, C_4) + \varepsilon(C_7, C_0) = 0$ $\varepsilon(C_2, C_3) + \varepsilon(C_6, C_7) + 2\varepsilon(C_3, C_4)\varepsilon(\overline{r_4}C_6, \overline{r_4}C_7) = 0$
$H_{\alpha_1+2\alpha_2,0}$	$\varepsilon(C_0, C_1) + \varepsilon(C_4, C_5) = 0$ $\varepsilon(C_3, C_4) + \varepsilon(C_7, C_0) = 0$ $\varepsilon(C_1, C_2) + \varepsilon(C_5, C_6) + 2\varepsilon(C_0, C_1)\varepsilon(\overline{r_1}C_1, \overline{r_1}C_2) = 0$

*Proof.* We begin with the following observation: for a given equation in the table, either all or



none of the corresponding hyperplanes are reducibility hyperplanes. We only need to prove each equation in the case where all  $\varepsilon$  are non-zero.

In order to prove this proposition, first, we need to discuss some results concerning the Weyl group. For  $s \in W$ , we have the following definitions (see 1.6 of [4]):

$$\begin{aligned}\Delta(s) &= \Delta^+ \cap s^{-1}(-\Delta) \\ n(s) &= \#\Delta(s)\end{aligned}$$

The product  $s = s_{i_1} \cdots s_{i_k} \in W$ , where  $s_{i_j} = s_{\alpha_{i_j}}$  and the  $\alpha_{i_j}$  are simple roots, is a *reduced expression* for  $s$  if  $k$  is minimal. The *length* of  $s$  is defined to be  $\ell(s) = k$ . We have  $\ell(s) = n(s) = \ell(s^{-1})$  (see Lemma 10.3 A of [3]). We note that  $\Delta(s) = \{s^{-1}(-\alpha) \mid \alpha \in \Delta^+ \text{ and } s^{-1}(-\alpha) > 0\}$ . We may rewrite this as  $\Delta(s) = \{\alpha \in \Delta^+ \mid s\alpha < 0\}$ . Also, if  $s = s_{i_1} \cdots s_{i_k}$  is a reduced expression for  $s \in W$ , then

$$\Delta(s^{-1}) = \{\alpha_{i_1}, s_{i_1}\alpha_{i_2}, \dots, s_{i_1} \cdots s_{i_{k-1}}\alpha_{i_k}\} \quad (5.2.1)$$

(see the proof of Corollary 1.7 of [4]).

**Claim 5.2.2.** *Recall that we defined the fundamental Weyl chamber  $\mathfrak{C}_0$  so that  $-\rho \in \mathfrak{C}_0$ . Let  $s \in W$  and  $\alpha \in \Delta^+$ . If the  $\alpha$  hyperplanes are positive in  $s\mathfrak{C}_0$ , then*

$$\#\{\beta \in \Delta^+ \mid \beta \text{ hyperplanes are positive in } s\mathfrak{C}_0\} > \#\{\beta \in \Delta^+ \mid \beta \text{ hyperplanes are positive in } s_\alpha s\mathfrak{C}_0\}.$$

*Proof.* Note that as

$$\begin{aligned}\{\beta \in \Delta^+ \mid \beta \text{ hyperplanes positive in } s\mathfrak{C}_0\} &= \{\beta \in \Delta^+ \mid (\beta, s(-\rho)) > 0\} \\ &= \{\beta \in \Delta^+ \mid s^{-1}\beta < 0\} \text{ by invariance of Killing form} \\ &= \Delta(s^{-1}) \text{ by definition,}\end{aligned}$$

we only need to show that  $\ell(s^{-1}) = \ell(s) > \ell(s_\alpha s) = \ell(s^{-1}s_\alpha)$  if the hypotheses for  $s$  and  $\alpha$  are satisfied. By (5.2.1), we may assume that  $\alpha = s_{i_1} \cdots s_{i_{j-1}}\alpha_{i_j}$  for some  $j \in \{1, \dots, k\}$ . Then  $s_\alpha = s_{i_1} \cdots s_{i_{j-1}}s_{i_j}s_{i_{j-1}} \cdots s_{i_1}$  by Proposition 1.2 of [4]. Therefore  $s_\alpha s = s_{i_1} \cdots s_{i_{j-1}}s_{i_{j+1}} \cdots s_{i_k}$ , whence  $\ell(s) > \ell(s_\alpha s)$ .  $\square$

**Type  $A_2$ :** In the following diagram, we label the alcove  $wC_0$  with  $w \in W_a$  and with  $T_\mu = \bar{w}^{-1}w$ , where  $T_\mu$  is translation by  $-\mu$ :  $T_\mu(\lambda) = \lambda - \mu$ .

$$\mathcal{C} = \{C_0, \dots, C_5\}$$

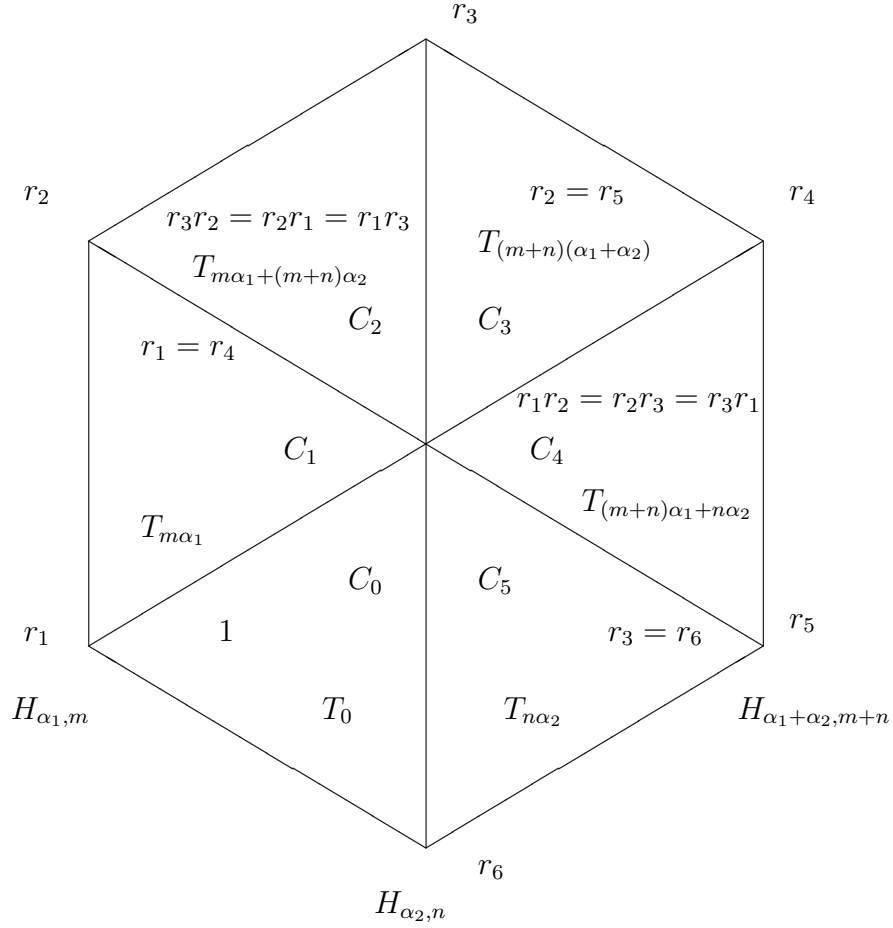


Figure 5-3: Type  $A_2$

**If  $m = 0$ :** The translations corresponding to alcoves are symmetric about the affine hyperplane  $H_{\alpha_1, m}$ :

$$\begin{aligned} T_0 &= T_{m\alpha_1} \\ T_{m\alpha_1+(m+n)\alpha_2} &= T_{n\alpha_2} \\ T_{(m+n)(\alpha_1+\alpha_2)} &= T_{(m+n)\alpha_1+n\alpha_2} \end{aligned}$$

Since we are interested in what happens when we cross reducibility hyperplanes, we may assume that  $n = m + n > 0$ . As  $C_0$  and  $s_{\alpha_1, m}C_0 = s_{\alpha_1}C_0$  are adjacent alcoves separated by a Weyl chamber wall (which is not a reducibility hyperplane), therefore  $\overline{w_I}^{-1}C_0$  and  $\overline{w_I}^{-1}s_{\alpha_1}C_0$  are adjacent alcoves separated by a Weyl chamber wall, whence

$$R^{\overline{w_I}^{-1}C_0} = R^{\overline{w_I}^{-1}s_{\alpha_1}C_0}.$$

As  $\overline{w_I}^{-1}w_I = \overline{w_J}^{-1}w_J \Rightarrow w_J = w_I$  or  $w_J = s_{\alpha_1}w_I$ , we conclude that (5.1.3) still holds. We consider that equation, for various values of  $\mu$ , and the subsets  $I$  which correspond to  $\mu$ :

$\mu = 0$ : As the translation is trivial, at least one of the hyperplanes associated with  $\varepsilon(I)$  is non-positive if  $I$  is non-empty, whence  $\varepsilon(I) = 0$  if  $I$  is non-empty.

$\mu = n\alpha_2$ : Subsets  $I$  corresponding to  $n\alpha_2$  are:  $\{3\}, \{6\}$ , and subsets of size at least 3.

Subsets  $I$  corresponding to  $m\alpha_1 + (m + n)\alpha_2$  are:  $\{2, 3\}, \{2, 6\}, \{5, 6\}$  (these correspond to  $r_3r_2$ );  $\{1, 2\}, \{1, 5\}, \{4, 5\}$  (these correspond to  $r_2r_1$ );  $\{4, 3\}$  (this corresponds to  $r_1r_3$ ); and subsets of size at least 4.

When we have the hyperplane  $H_{\alpha, n}$  where  $\alpha$  and  $n$  are positive separating adjacent alcoves  $wA_0$  and  $w'A_0$ , recall that

$$R^{wA_0}(\lambda) = R^{w'A_0}(\lambda) + 2\varepsilon(wA_0, w'A_0)R^{\overline{s_{\alpha, n}w'}}(\overline{s_{\alpha, n}}s_{\alpha, n}\lambda).$$

If  $w'A_0$  lies in the interior of the Weyl chamber  $s\mathfrak{C}_0$ , then  $s_{\alpha}w'A_0$  lies in  $s_{\alpha}s\mathfrak{C}_0$ . By our claim, the number of positive roots with corresponding hyperplanes positive in  $s\mathfrak{C}_0$  is greater than the number for  $s_{\alpha}s\mathfrak{C}_0$ . As two of the three hyperplanes of  $\mathcal{C}$  are positive in the case where  $m = 0$  and  $n > 0$ , we cannot have three or more positive hyperplanes corresponding to  $\varepsilon(I)$ . As the number of hyperplanes corresponding to  $\varepsilon(I)$  is  $|I|$ , we conclude that  $\varepsilon(I) = 0$  if  $|I| \geq 3$ .

If 1 or 4 belongs to  $I$ , then  $\varepsilon(I) = 0$ . This is because  $ws_{\alpha, k}w^{-1} = s_{w\alpha, k}$ , and because  $r_1$  and  $r_4$  correspond to reflection through  $H_{\alpha_1, 0}$ , which is not a reducibility hyperplane.

The affine reflections corresponding to  $I = \{2, 3\}$  are  $r_2$  and  $\overline{r_2}r_3\overline{r_2}$ , which cor-

respond to the hyperplanes  $H_{\alpha_1+\alpha_2, m+n}$  and  $H_{s_{\alpha_1+\alpha_2}(\alpha_2), n} = H_{\alpha_1, -n}$ , respectively. As the second hyperplane is not a reducibility hyperplane, therefore  $\varepsilon(\{2, 3\}) = 0$ . Similarly,  $\varepsilon(\{2, 6\}) = \varepsilon(\{5, 6\}) = 0$ .

Thus (5.1.3) for  $\mu = n\alpha_2$  gives us:

$$\varepsilon(C_2, C_3) + \varepsilon(C_5, C_0) = 0.$$

We will provide less detail in subsequent cases. The arguments are similar. It is helpful to refer to Figure 5.2.

$\mu = (m+n)(\alpha_1 + \alpha_2)$ : Again, if  $|I| \geq 3$ , then  $\varepsilon(I) = 0$ . We have:

$I$	Corresponding $w_I$	Corresponding $\overline{w_I}^{-1}w_I$
$\{2\}, \{5\}$	$r_2$	$T_{(m+n)(\alpha_1+\alpha_2)}$
$\{2, 4\}$	$r_1 r_2$	$T_{(m+n)\alpha_1+n\alpha_2}$
$\{1, 3\}, \{1, 6\}, \{4, 6\}$	$r_3 r_1$	
$\{3, 5\}$	$r_2 r_3$	

As before, if  $I$  contains 1 or 4, then  $\varepsilon(I) = 0$ . The hyperplanes corresponding to  $\{3, 5\}$  are  $H_{\alpha_2, n}$  and  $H_{\alpha_1, m+n}$ , which are reducibility hyperplanes. We conclude from (5.1.3) that

$$\varepsilon(C_1, C_2) + \varepsilon(C_4, C_5) + 2\varepsilon(C_2, C_3)\varepsilon(\overline{r_3}C_4, \overline{r_3}C_5) = 0.$$

**If  $n = 0$ :** Symmetry with the case  $m = 0$  gives:

$$\begin{aligned} \varepsilon(C_0, C_1) + \varepsilon(C_3, C_4) &= 0 \quad \text{and} \\ \varepsilon(C_5, C_4) + \varepsilon(C_2, C_1) + 2\varepsilon(C_4, C_3)\varepsilon(\overline{r_4}C_2, \overline{r_4}C_1) &= 0 \\ \iff \varepsilon(C_1, C_2) + \varepsilon(C_4, C_5) + 2\varepsilon(C_0, C_1)\varepsilon(\overline{r_1}C_1, \overline{r_1}C_2) &= 0 \end{aligned}$$

**If  $m + n = 0$ :** As we are interested in what happens when we cross reducibility hyperplanes, we may assume that  $m$  and  $n$  are non-zero. Without loss of generality, assume  $m > 0$

and  $n < 0$ .

For any  $I$  corresponding to  $T_0, T_{(m+n)(\alpha_1+\alpha_2)}, T_{m\alpha_1}$ , or  $T_{m\alpha_1+(m+n)\alpha_2}$ ,  $\varepsilon(I) = 0$  if  $I$  is non-empty as the translation is by a non-positive amount so that at least one of the associated hyperplanes must be non-positive. Since  $\mathcal{C}$  contains only one positive hyperplane, by Claim 5.2.2,  $\varepsilon(I) = 0$  whenever  $|I| \geq 2$ . From (5.1.2), we conclude that

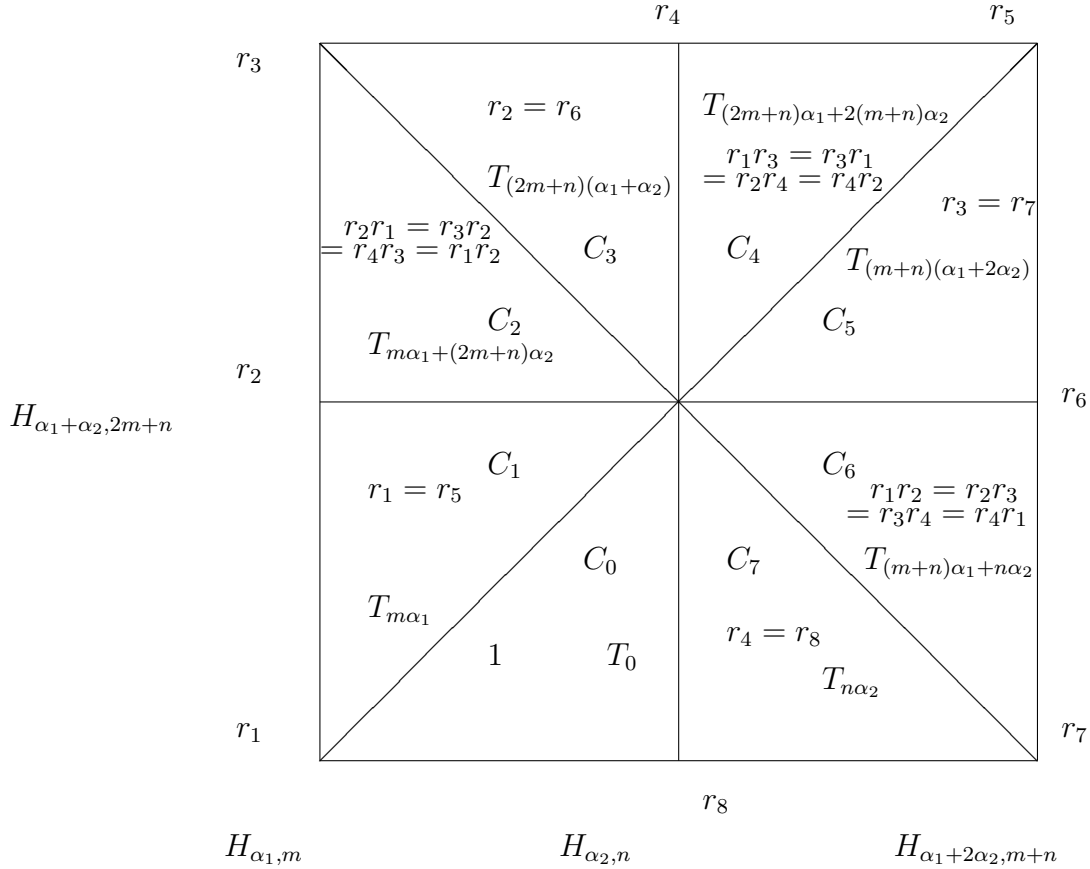
$$\varepsilon(C_2, C_3) + \varepsilon(C_5, C_0) = 0.$$

Symmetrically, if  $m > 0$  and  $n < 0$ ,

$$\varepsilon(C_0, C_1) + \varepsilon(C_3, C_4) = 0.$$

**Type  $B_2$ :** We label the diagram as we did for type  $A_2$ :

$$\mathcal{C} = \{C_0, \dots, C_7\}$$



**If  $m = 0$ :** We may assume  $n > 0$ . Since  $C_0$  and  $s_{\alpha_1, m}C_0$  are adjacent alcoves separated by a Weyl chamber wall, as before, (5.1.3) holds. We examine this equation for different values of  $\mu$ :

$\mu = 0$ : As before,  $\varepsilon(I) = 0$  for non-empty  $I$ .

$\mu = n\alpha_2$ : For now, restrict our attention to  $I$  of size less than three. We have:

$I$	Corresponding $w_I$	Corresponding $\overline{w_I}^{-1}w_I$
$\{4\}, \{8\}$	$r_4$	$T_{n\alpha_2}$
$\{1, 2\}, \{1, 6\}, \{5, 6\}$	$r_2r_1$	$T_{m\alpha_1 + (2m+n)\alpha_2}$
$\{2, 3\}, \{2, 7\}, \{6, 7\}$	$r_3r_2$	
$\{3, 4\}, \{3, 8\}, \{7, 8\}$	$r_4r_3$	
$\{4, 5\}$	$r_1r_4$	

Note that  $\overline{r_2}r_3\overline{r_2}$  and  $\overline{r_3}r_4\overline{r_3}$  correspond to  $H_{\alpha_1, -(m+n)}$  and  $H_{\alpha_1 + \alpha_2, -n}$ , respectively, and therefore  $\varepsilon(I) = 0$  for  $I$  corresponding to  $r_3r_2$  and  $r_4r_3$ . As  $r_1$  and  $r_5$  correspond to reflection through a Weyl chamber wall,  $\varepsilon(I) = 0$  for  $I$  containing 1 or 5. Therefore, if we take (5.1.3) modulo 8, we obtain

$$\varepsilon(C_3, C_4) + \varepsilon(C_7, C_0) = 0.$$

$\mu = (2m + n)(\alpha_1 + \alpha_2)$ : If  $|I| \geq 4$ , then  $\varepsilon(I) = 0$  by Claim 5.2.2. Consider  $I$  of size less than four:

$I$	Corresponding $w_I$	Corresponding $\overline{w_I}^{-1}w_I$
$\{2\}, \{6\}$	$r_2$	$T_{(2m+n)(\alpha_1 + \alpha_2)}$
$I = \{i_1 < i_2 < i_3\}$	$r_{i_3}r_{i_2}r_{i_1} = r_2$	
$\{2, 5\}$	$r_1r_2$	$T_{(m+n)\alpha_1 + n\alpha_2}$
$\{3, 6\}$	$r_2r_3$	
$\{4, 7\}$	$r_3r_4$	
$\{1, 4\}, \{1, 8\}, \{5, 8\}$	$r_4r_1$	

Observe that  $\overline{r_3}r_2\overline{r_3}$  corresponds to  $H_{\alpha_2, -(2m+n)}$ , and so  $\varepsilon(I) = 0$  for  $I$  corresponding to  $r_2r_3$ . Meanwhile,  $r_4$  and  $\overline{r_4}r_3\overline{r_4}$  correspond to  $H_{\alpha_2, n}$  and  $H_{\alpha_1, m+n}$ , respectively. We conclude that  $\varepsilon(\{4, 7\}) \neq 0$ .

If  $I$  contains 1 or 5, then  $\varepsilon(I) = 0$ . The only subsets  $I = \{i_1 < i_2 < i_3\}$  which do not contain 1 or 5 for which  $r_2 = r_{i_3}r_{i_2}r_{i_1}$  are:  $I = \{3, 4, 6\}, \{4, 6, 8\}$ . (Note that computations may be done using the relations listed in the diagram.) As noted previously,  $\overline{r_3}r_4\overline{r_3}$  corresponds to  $H_{\alpha_1 + \alpha_2, -n}$ , and so  $\varepsilon(\{3, 4, 6\}) = 0$ . Note that  $\overline{r_4}r_6\overline{r_8}r_6\overline{r_4} = \overline{r_4}r_4\overline{r_4} = s_{\alpha_2, -n}$  as  $r_6$  and  $r_8$  correspond to roots that are orthogonal to each other and  $r_4 = r_8$ . Therefore,  $\varepsilon(\{4, 6, 8\}) = 0$ .

Thus (5.1.3) gives

$$\varepsilon(C_1, C_2) + \varepsilon(C_5, C_6) + 2\varepsilon(C_3, C_4)\varepsilon(\overline{r_4}C_6, \overline{r_4}C_7) = 0.$$

$\mu = (m+n)(\alpha_1 + 2\alpha_2)$ : Consider  $I$  of size less than three:

$I$	Corresponding $w_I$	Corresponding $\overline{w_I}^{-1}w_I$
$\{3\}, \{7\}$	$r_3$	$T_{(m+n)(\alpha_1 + 2\alpha_2)}$
$\{1, 3\}, \{1, 7\}, \{5, 7\}, \{3, 5\}$	$r_3r_1, r_1r_3$	$T_{(2m+n)\alpha_1 + 2(m+n)\alpha_2}$
$\{2, 4\}, \{2, 8\}, \{6, 8\}, \{4, 6\}$	$r_4r_2, r_4r_2$	

If  $I$  contains 1 or 5, then  $\varepsilon(I) = 0$ . For each of  $I = \{2, 4\}, \{2, 8\}, \{6, 8\}$ , and  $\{4, 6\}$ ,  $\varepsilon(I) \neq 0$  as  $r_2$  and  $r_4$  correspond to reducibility hyperplanes, and since the roots corresponding to  $r_2$  and  $r_4$  are orthogonal. Note that there is an *even* number of such  $I$ . Thus, (5.1.3) taken modulo 8 gives

$$\varepsilon(C_2, C_3) + \varepsilon(C_6, C_7) = 0.$$

**If  $n = 0$ :** Since  $C_0$  and  $s_{\alpha_2, n}C_0$  are adjacent alcoves separated by a Weyl chamber wall, as before, (5.1.3) holds.

$\mu = m\alpha_1$ : Consider  $I$  of size less than three:

$I$	Corresponding $w_I$	Corresponding $\overline{w_I}^{-1}w_I$
$\{1\}, \{5\}$	$r_1$	$T_{m\alpha_1}$
$\{2, 5\}$ $\{3, 6\}$ $4 \text{ or } 8 \in I$	$r_1 r_2$ $r_2 r_3$ $r_3 r_4, r_4 r_1$	$T_{(m+n)\alpha_1+n\alpha_2}$

Since  $\overline{r_2 r_1 r_2}$  and  $\overline{r_3 r_2 r_3}$  correspond to  $H_{\alpha_1+2\alpha_2, -m}$  and  $H_{\alpha_2, -(2m+n)}$  respectively and since  $\varepsilon(I) = 0$  for  $I$  containing 4 or 8, therefore taking (5.1.3) modulo 8, we obtain

$$\varepsilon(C_0, C_1) + \varepsilon(C_4, C_5) = 0. \quad (5.2.2)$$

$\mu = (2m+n)(\alpha_1 + \alpha_2)$ : Consider  $I$  of size less than three:

$I$	Corresponding $w_I$	Corresponding $\overline{w_I}^{-1}w_I$
$\{2\}, \{6\}$	$r_2$	$T_{(2m+n)(\alpha_1+\alpha_2)}$
$\{1, 3\}, \{1, 7\}, \{5, 7\}, \{3, 5\}$ $4 \text{ or } 8 \in I$	$r_1 r_3, r_3 r_1$ $r_2 r_4, r_4 r_2$	$T_{(2m+n)\alpha_1+2(m+n)\alpha_2}$

Recall that  $\varepsilon(I) = 0$  for  $I$  containing 4 or 8.

As the roots which correspond to  $r_1$  and  $r_3$  are orthogonal and since the corresponding hyperplanes are reducibility hyperplanes, therefore  $\varepsilon(I) \neq 0$  for each of the *four*  $I$  corresponding to  $r_1 r_3$  and  $r_3 r_1$ . Thus, if we take (5.1.3) modulo 8, we obtain

$$\varepsilon(C_1, C_2) + \varepsilon(C_5, C_6) = 0.$$

$\mu = (m+n)(\alpha_1 + 2\alpha_2)$ : By Claim 5.2.2, we only need to consider  $I$  of size less than



four:

$I$	Corresponding $w_I$	Corresponding $\overline{w_I}^{-1}w_I$
$\{3\}, \{7\}$	$r_3$	$T_{(m+n)(\alpha_1+2\alpha_2)}$
$I = \{i_1 < i_2 < i_3\}$	$r_{i_3}r_{i_2}r_{i_1} = r_3$	
$\{1, 2\}, \{1, 6\}, \{5, 6\}$	$r_2r_1$	$T_{m\alpha_1+(2m+n)\alpha_2}$
$\{2, 3\}, \{2, 7\}, \{6, 7\}$	$r_3r_2$	
$4 \text{ or } 8 \in I$	$r_4r_3, r_1r_4$	

As  $\overline{r_2}r_3\overline{r_2}$  corresponds to  $H_{\alpha_1, -(m+n)}$ , therefore  $\varepsilon(I) = 0$  for  $I$  corresponding to  $r_3r_2$ .

Note that  $\overline{r_1}r_2\overline{r_1}$  corresponds to  $H_{\alpha_2, 2m+n}$ , which is a reducibility hyperplane. From (5.2.2),  $\varepsilon(\{1, 6\}) + \varepsilon(\{5, 6\}) = 0$ .

As before, if  $4 \text{ or } 8 \in I$ , then  $\varepsilon(I) = 0$ .

The only possible subsets  $I = \{i_1 < i_2 < i_3\}$  which do not contain 4 or 8 for which  $r_{i_3}r_{i_2}r_{i_1} = r_3$  are  $I = \{2, 5, 6\}$  and  $\{1, 3, 5\}$ . Orthogonality arguments give  $\varepsilon(\{1, 3, 5\}) = 0$ . As  $\overline{r_2}r_1\overline{r_2}$  corresponds to  $H_{\alpha_1+2\alpha_2, -m}$ , therefore  $\varepsilon(\{2, 5, 6\}) = 0$ .

From (5.1.3), we conclude that

$$\varepsilon(C_2, C_3) + \varepsilon(C_6, C_7) + 2\varepsilon(C_0, C_1)\varepsilon(\overline{r_1}C_1, \overline{r_1}C_2) = 0.$$

**If  $2m + n = 0$ :** We may assume that  $m$  and  $n$  are non-zero.

**If  $m > 0, n < 0$ :** For non-positive  $\mu$  and for non-empty  $I$  corresponding to  $T_\mu$ ,  $\varepsilon(I) = 0$ .

Therefore, for  $I$  corresponding to  $T_0$ ,  $T_{(2m+n)(\alpha_1+\alpha_2)}$ ,  $T_{n\alpha_2}$ ,  $T_{(2m+n)\alpha_1+2(m+n)\alpha_2}$ ,  $T_{(m+n)(\alpha_1+2\alpha_2)}$ , and  $T_{(m+n)\alpha_1+n\alpha_2}$ ,  $\varepsilon(I) = 0$  if  $I$  is non-empty. Thus equation (5.1.2) reduces to

$$\sum_{\substack{\emptyset \neq I \subset \{1, \dots, \ell\} \\ \overline{w_I}^{-1}w_I = m\alpha_1}} 2^{|I|} \varepsilon(I) R^{\overline{w_I}^{-1}C_0} (\lambda - m\alpha_1) = 0$$

As  $\mathcal{C}$  only contains one positive hyperplane,  $H_{\alpha_1, m}$ , we conclude that  $\varepsilon(I) = 0$  for  $|I| \geq 2$ . Furthermore, for any  $I$  in the above sum for which  $\varepsilon(I) \neq 0$ ,  $\overline{w_I}^{-1}C_0$  lies in

the Wallach region. From this, we conclude that

$$\varepsilon(C_0, C_1) + \varepsilon(C_4, C_5) = 0.$$

**If  $m < 0, n > 0$ :** All  $\varepsilon(I)$  are zero for non-empty  $I$  corresponding to  $T_0$ ,  $T_{(2m+n)(\alpha_1+\alpha_2)}$ ,  $T_{m\alpha_1}$ , and  $T_{m\alpha_1+(2m+n)\alpha_2}$ . Thus, we may rewrite (5.1.2) as

$$\sum_{\substack{\emptyset \neq I \subset \{1, \dots, \ell\} \\ \overline{w_I}^{-1} w_I = \mu_1}} 2^{|I|} \varepsilon(I) R^{\overline{w_I}^{-1} C_0} (\lambda - \mu_1) + \sum_{\substack{\emptyset \neq J \subset \{1, \dots, \ell\} \\ \overline{w_J}^{-1} w_J = \mu_2}} 2^{|J|} \varepsilon(J) R^{\overline{w_J}^{-1} C_0} (\lambda - \mu_2) = 0$$

where  $\mu_1 = n\alpha_2$  and  $\mu_2 = (m+n)(\alpha_1 + 2\alpha_2)$ . Consider the first sum.

As  $\mathcal{C}$  contains two positive hyperplanes, for  $I$  of size greater than two,  $\varepsilon(I) = 0$ .

Restricting our attention to  $|I| \leq 2$ , we get:

$I$	Corresponding $w_I$	Corresponding $\overline{w_I}^{-1} w_I$
$\{4\}, \{8\}$	$r_4$	$T_{n\alpha_2}$
$2 \text{ or } 6 \in I$	$r_1 r_3, r_3 r_1$ $r_2 r_4, r_4 r_2$	$T_{(2m+n)\alpha_1 + 2(m+n)\alpha_2}$

If  $I$  contains 2 or 6, then  $\varepsilon(I) = 0$ , as  $r_2$  corresponds to reflection through a Weyl chamber wall.

As the hyperplane corresponding to  $r_1$  is not a reducibility hyperplane and since the roots corresponding to  $r_1$  and  $r_3$  are orthogonal, therefore  $\varepsilon(I) = 0$  for  $I$  corresponding to  $r_1 r_3$  and  $r_3 r_1$ .

As  $n\alpha_2$  is strictly smaller than  $(m+n)(\alpha_1 + 2\alpha_2) = (m+n)\alpha_1 + n\alpha_2$  in the partial ordering on  $\Lambda$ , by our arguments above,

$$\varepsilon(C_3, C_4) + \varepsilon(C_7, C_0) = 0$$

and in fact, each summation must be zero.

Now we consider the second sum. Again, we restrict our attention to  $I$  of size no

more than two:

$I$	Corresponding $w_I$	Corresponding $\overline{w_I}^{-1}w_I$
$\{3\}, \{7\}$	$r_3$	$T_{(m+n)(\alpha_1+2\alpha_2)}$
$2 \text{ or } 6 \in I$	$r_1r_2, r_2r_3$	$T_{(m+n)\alpha_1+n\alpha_2}$
$\{4, 7\}$	$r_3r_4$	
$\{1, 4\}, \{1, 8\}, \{5, 8\}$	$r_4r_1$	

The reflection  $r_1$  corresponds to the hyperplane  $H_{\alpha_1, m}$  which is not a reducibility hyperplane. We conclude that  $\varepsilon(I) = 0$  for  $I$  corresponding to  $r_4r_1$ .

As  $r_4$  corresponds to  $H_{\alpha_2, n}$  and as  $\overline{r_4}r_3\overline{r_4}$  corresponds to  $H_{\alpha_1, m+n}$ , both of which are reducibility hyperplanes, therefore  $\varepsilon(\{4, 7\}) \neq 0$ .

Recall that if  $I$  contains 2 or 6, then  $\varepsilon(I) = 0$ .

Combining our results, and observing that  $\overline{w_I}^{-1}\mathcal{C}$  lies in the Wallach region for  $I = \{3\}, \{7\}$ , and  $\{4, 7\}$ , because our second sum must equal zero, we obtain

$$\varepsilon(C_2, C_3) + \varepsilon(C_6, C_7) + 2\varepsilon(C_3, C_4)\varepsilon(\overline{r_4}C_6, \overline{r_4}C_7) = 0.$$

**If  $m + n = 0$ :** We may assume that  $m$  and  $n$  are non-zero. We consider the following two cases:

**If  $m < 0, n > 0$ :** For  $I$  corresponding to  $T_0, T_{(m+n)(\alpha_1+2\alpha_2)}, T_{m\alpha_1}, T_{(2m+n)\alpha_1+2(m+n)\alpha_2}, T_{(2m+n)(\alpha_1+\alpha_2)}$ , and  $T_{m\alpha_1+(2m+n)\alpha_2}$ ,  $\varepsilon(I) = 0$  if  $I$  is non-empty. Arguing as in the case  $2m + n = 0$  with  $m > 0$  and  $n < 0$ , we get

$$\sum_{\substack{\emptyset \neq I \subset \{1, \dots, \ell\} \\ \overline{w_I}^{-1}w_I = n\alpha_2}} 2^{|I|} \varepsilon(I) = 0$$

$$\text{so} \quad \varepsilon(C_3, C_4) + \varepsilon(C_7, C_0) = 0.$$

**If  $m > 0, n < 0$ :** All  $\varepsilon(I)$  are zero for non-empty  $I$  corresponding to  $T_0, T_{(m+n)(\alpha_1+2\alpha_2)},$

$T_{n\alpha_2}$ , and  $T_{(m+n)\alpha_1+n\alpha_2}$ . Thus, we may rewrite (5.1.2) as

$$\sum_{\substack{\emptyset \neq I \subset \{1, \dots, \ell\} \\ \overline{w_I}^{-1} w_I = \mu_1}} 2^{|I|} \varepsilon(I) R^{\overline{w_I}^{-1} C_0} (\lambda - \mu_1) + \sum_{\substack{\emptyset \neq J \subset \{1, \dots, \ell\} \\ \overline{w_J}^{-1} w_J = \mu_2}} 2^{|J|} \varepsilon(J) R^{\overline{w_J}^{-1} C_0} (\lambda - \mu_2) = 0$$

where  $\mu_1 = m\alpha_1$  and  $\mu_2 = (2m+n)(\alpha_1 + \alpha_2)$ . Consider the first sum and the  $I$  of size no more than two in that summation (as  $\mathcal{C}$  contains two positive hyperplanes):

$I$	Corresponding $w_I$	Corresponding $\overline{w_I}^{-1} w_I$
$\{1\}, \{5\}$	$r_1$	$T_{m\alpha_1}$
$3 \text{ or } 7 \in I$	$r_1 r_3, r_3 r_1$ $r_2 r_4, r_4 r_2$	$T_{(2m+n)\alpha_1 + 2(m+n)\alpha_2}$

Arguing as for the first sum in the case where  $2m+n=0$ ,  $m < 0$ , and  $n > 0$ , we conclude that  $\varepsilon(I) = 0$  for  $I$  corresponding to  $r_1 r_3$ ,  $r_3 r_1$ ,  $r_2 r_4$ , and  $r_4 r_2$ . We conclude that

$$\varepsilon(C_0, C_1) + \varepsilon(C_4, C_5) = 0 \quad (5.2.3)$$

and as before, each summation must equal zero.

Now we consider the second sum. Again, we restrict our attention to  $I$  of size no more than two:

$I$	Corresponding $w_I$	Corresponding $\overline{w_I}^{-1} w_I$
$\{2\}, \{6\}$	$r_2$	$T_{(2m+n)(\alpha_1 + \alpha_2)}$
$\{1, 2\}, \{1, 6\}, \{5, 6\}$	$r_2 r_1$	$T_{m\alpha_1 + (2m+n)\alpha_2}$
$3 \text{ or } 7 \in I$	$r_3 r_2, r_4 r_3$	
$\{4, 5\}$	$r_1 r_4$	

As  $r_4$  corresponds to a negative hyperplane, therefore  $\varepsilon(\{4, 5\}) = 0$ .

If  $I$  contains 3 or 7, then  $\varepsilon(I) = 0$ .

The reflections  $r_1$  and  $\overline{r_1} r_2 \overline{r_1}$  correspond to the hyperplanes  $H_{\alpha_1, m}$  and  $H_{\alpha_2, 2m+n}$ ,

respectively, which are reducibility hyperplanes. By (5.2.3),  $\varepsilon(\{1, 6\}) + \varepsilon(\{5, 6\}) = 0$ . Combining our results and observing that  $\overline{w_I}^{-1}\mathcal{C}$  lies in the Wallach region for  $I = \{2\}, \{6\}$ , and  $\{1, 2\}$ , because our second sum must equal zero, we obtain

$$\varepsilon(C_1, C_2) + \varepsilon(C_5, C_6) + 2\varepsilon(C_0, C_1)\varepsilon(\overline{r_1}C_1, \overline{r_1}C_2) = 0.$$

This ends the proof of the proposition. □

**Remark 5.2.3.** *Observe that our computations show that when crossing a hyperplane corresponding to  $\alpha_1$  or  $\alpha_2$ , the value of  $\varepsilon$  does not depend on the Weyl chamber containing the point of crossing. Furthermore, none of our arguments referred to simplicity of the  $\alpha_i$ .*

### 5.3 Using induction to obtain the general case

**Definition 5.3.1.** Fix a hyperplane  $H_{\gamma,N}$  and  $s \in W$ . We let  $\varepsilon(H_{\gamma,N}, s)$  be the value of any  $\varepsilon(wA_0, w'A_0)$ , where  $H_{\gamma,N}$  separates the adjacent alcoves  $wA_0$  and  $w'A_0$ ,  $wA_0 \subset H_{\gamma,N}^+$  and  $w'A_0 \subset H_{\gamma,N}^-$ , and  $wA_0 \subset s\mathfrak{C}_0$  (and hence  $w'A_0 \subset s\mathfrak{C}_0$ ). By Proposition 5.1.1, this is well-defined.

We begin by computing  $\varepsilon$  for a simple root  $\alpha$ .

**Lemma 5.3.2.** Let  $\delta_\alpha$  be  $-1$  if  $\alpha$  is noncompact, and  $1$  if it is compact. If  $\alpha$  is simple and  $n$  is positive, then  $\varepsilon(H_{\alpha,n}, s) = \delta_\alpha^n$ .

*Proof.* Choose a standard triple  $X_\alpha \in \mathfrak{g}_\alpha$ ,  $Y_\alpha \in \mathfrak{g}_{-\alpha}$ , and  $H_\alpha = [X_\alpha, Y_\alpha] \in \mathfrak{h}$  satisfying  $\mu(H_\alpha) = (\mu, \alpha^\vee) \forall \mu \in \mathfrak{h}^*$ . We have the relations

$$[H_\alpha, X_\alpha] = 2X_\alpha, \quad [H_\alpha, Y_\alpha] = -2Y_\alpha, \quad [X_\alpha, Y_\alpha] = H_\alpha,$$

$$\alpha(H_\alpha) = (\alpha, \alpha^\vee) = 2.$$

Taking complex conjugates, multiplying by  $-1$ , and using anti-commutativity,

$$[-\bar{H}_\alpha, \bar{X}_\alpha] = -2\bar{X}_\alpha, \quad [-\bar{H}_\alpha, \bar{Y}_\alpha] = 2\bar{Y}_\alpha, \quad [\bar{Y}_\alpha, \bar{X}_\alpha] = -\bar{H}_\alpha,$$

$$\bar{\alpha}(\bar{H}_\alpha) = (\bar{\alpha}, \bar{\alpha}^\vee) = 2.$$

If  $\alpha$  is imaginary, then  $\bar{X}_\alpha \in \mathfrak{g}_{-\alpha}$  and  $\bar{Y}_\alpha \in \mathfrak{g}_\alpha$ . Also,  $-\bar{H}_\alpha = H_\alpha$ . The above relations give  $(\bar{Y}_\alpha, \bar{X}_\alpha, -\bar{H}_\alpha) = (cX_\alpha, c^{-1}Y_\alpha, H_\alpha)$  for some non-zero scalar  $c$ .  $B(X, \bar{X})$  is positive for non-zero  $X \in \mathfrak{p}$  and negative for non-zero  $X \in \mathfrak{k}$ . By Lemma 2.18a) of [7], if  $\alpha$  is compact, then  $c < 0$  and if  $\alpha$  is noncompact, then  $c > 0$ . We may arrange for  $c$  to be  $\pm 1$ . We have:

$$-\bar{Y}_\alpha = \delta_\alpha X_\alpha.$$

The  $\lambda - n\alpha$  weight space of  $M(\lambda)$  is one-dimensional and spanned by the vector  $Y_\alpha^n v_\lambda$ . We

know that

$$\begin{aligned}\langle Y_\alpha^n v_\lambda, Y_\alpha^n v_\lambda \rangle_\lambda &= \delta_\alpha^n \langle v_\lambda, X_\alpha^n Y_\alpha^n v_\lambda \rangle_\lambda \\ &= \delta_\alpha^n n! \langle v_\lambda, H_\alpha(H_\alpha - 1) \cdots (H_\alpha - (n-1)) v_\lambda \rangle_\lambda\end{aligned}$$

from  $\mathfrak{sl}_2$  theory. As  $\lambda(H_\alpha) - j$  is positive for  $j < n-1$  and  $\lambda \in H_{\alpha,n}^- \cap H_{\alpha,n-1}^+$ , negative for  $j = n-1$  and  $\lambda \in H_{\alpha,n}^- \cap H_{\alpha,n-1}^+$ , while it is positive for  $j = n-1$  and  $\lambda \in H_{\alpha,n}^+$ , we conclude that  $\varepsilon(H_{\alpha,n}, s) = \delta_\alpha^n$ .  $\square$

**Lemma 5.3.3.** *Let  $\gamma$  be a positive non-simple root. There exists some simple root  $\alpha$  such that  $(\gamma, \alpha) > 0$  and  $s_\alpha \gamma > 0$ .*

*Proof.* The first statement follows from Lemma 10.2A of [3] and the second from Lemma 10.2B.  $\square$

**Proposition 5.3.4.** *Let  $\gamma$  be a positive non-simple root. Let  $\alpha$  and  $\beta = s_\alpha \gamma$  be the roots provided by Lemma 5.3.3. If  $\alpha, \gamma$  do not generate a type  $G_2$  root system, then:*

*If  $|\gamma| = |\alpha|$ :*

$$\varepsilon(H_{\gamma,N}, s) = \begin{cases} -\delta_\alpha^N \varepsilon(H_{\beta,N}, s_\alpha s) & \text{if } \alpha \text{ and } \beta \text{ hyperplanes are positive on } s\mathfrak{C}_0 \\ \delta_\alpha^N \varepsilon(H_{\beta,N}, s_\alpha s) & \text{otherwise.} \end{cases}$$

*If  $2|\gamma|^2 = |\alpha|^2$ :*

$$\varepsilon(H_{\gamma,N}, s) = \begin{cases} -\delta_\alpha^N \varepsilon(H_{\beta,N}, s_\alpha s) & \text{if } \alpha \text{ and } \alpha + 2\beta = s_\beta \alpha \text{ hyperplanes are positive on } s\mathfrak{C}_0 \\ \delta_\alpha^N \varepsilon(H_{\beta,N}, s_\alpha s) & \text{otherwise.} \end{cases}$$

*If  $|\gamma|^2 = 2|\alpha|^2$ :*

$$\varepsilon(H_{\gamma,N}, s) = \begin{cases} \varepsilon(H_{\beta,N}, s_\alpha s) & \text{if } \alpha \text{ and } \alpha + \beta = s_\beta \alpha \text{ hyperplanes are positive on } s\mathfrak{C}_0 \\ -\varepsilon(H_{\beta,N}, s_\alpha s) & \text{otherwise.} \end{cases}$$

*Proof.* Consider a two-dimensional slice  $P = \text{span}\{\alpha, \gamma\} + \mu_0$  through  $s\mathfrak{C}_0$ , where  $\mu_0$  lies in the intersection of  $H_{\gamma,N}$  and  $H_{\alpha,k}$  for some integer  $k$ , and  $(\mu_0, \delta^\vee)$  is not an integer for any root  $\delta$  that does not lie in the root subsystem generated by  $\alpha$  and  $\gamma$ . We are in the leftmost situation of Figure 5-2. If we take a suitably small circular path around  $\mu_0$  in  $P$ , due to Remark 5.2.3, the proof of

Proposition 5.2.1 still applies with  $\alpha$  and  $\gamma$  corresponding to a suitable choice of the roots in the root system generated by  $\alpha_1$  and  $\alpha_2$ . Further, our observation in Remark 5.2.3 still holds, so that for  $\alpha$  and some root  $\delta$  in our root system generated by  $\alpha$  and  $\gamma$  corresponding to the  $\alpha_i$ , the values for  $\varepsilon$  for the hyperplanes  $H_{\alpha,k}$  and  $H_{\delta,k}$  do not change as we cross Weyl chamber walls along a path restricted to  $P$ .

**Case  $|\gamma| = |\alpha|$ :** First, we examine the rank 2 case using the setup of Figure 5-1 when  $m = 0$ . Our equation from Proposition 5.2.1 gives:

$$\begin{array}{ccccccc}
- & + & & + & - & & - & + & & + & - \\
\varepsilon(C_1, C_2) & + & \varepsilon(C_4, C_5) & + & 2\varepsilon(C_2, C_3)\varepsilon(\overline{r_3}C_4, \overline{r_3}C_5) & = & 0 \\
H_{\alpha_1+\alpha_2, m+n} & & H_{\alpha_1+\alpha_2, m+n} & & H_{\alpha_2, n} & & H_{\alpha_1, m+n}
\end{array}$$

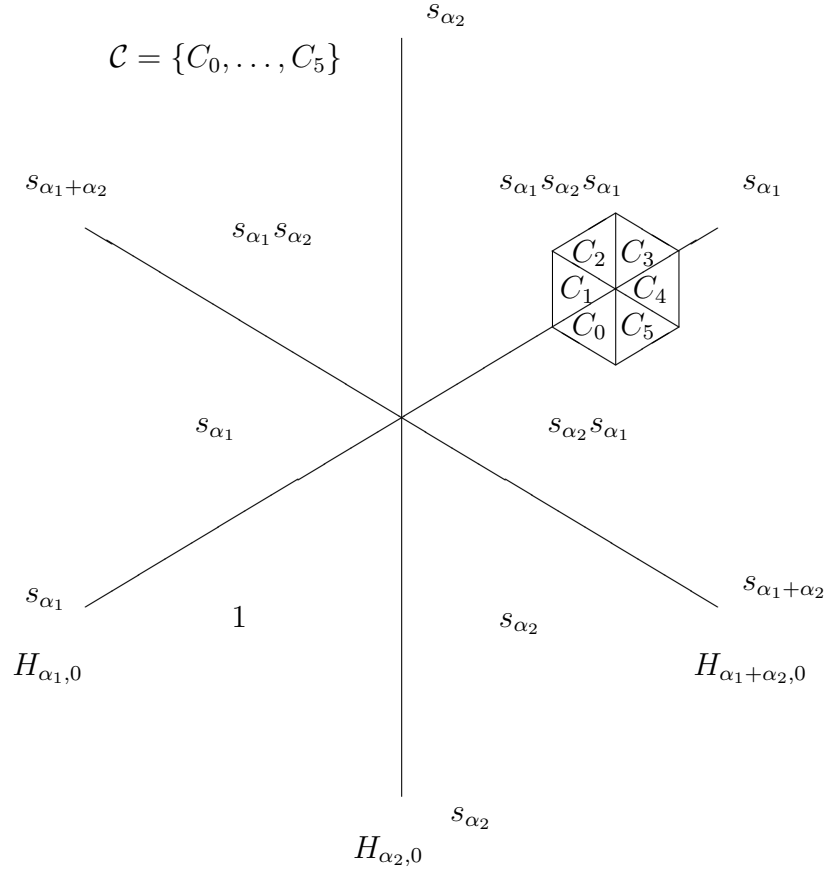


Figure 5-4: Type  $A_2$ : calculating  $\varepsilon$  for  $H_{\alpha_1+\alpha_2,N}$



$$\begin{aligned}
\varepsilon(H_{\alpha_1+\alpha_2,N}, s_{\alpha_2}s_{\alpha_1}) &= -\varepsilon(H_{\alpha_1+\alpha_2,N}, s_{\alpha_1}s_{\alpha_2}s_{\alpha_1}) &= \varepsilon(H_{\alpha_2,N}, s_{\alpha_1}s_{\alpha_2}s_{\alpha_1})\varepsilon(H_{\alpha_1,N}, s_{\alpha_1}) \\
&\text{by Lemma 5.3.2} &= \delta_{\alpha_1}^N \varepsilon(H_{\alpha_2,N}, s_{\alpha_1}s_{\alpha_2}s_{\alpha_1}).
\end{aligned}$$

Proposition 5.2.1 indicates that  $\varepsilon(H_{\alpha_1+\alpha_2,N}, s)$  changes sign as we cross the hyperplane  $H_{\alpha_2,0}$ , so we also have

$$\varepsilon(H_{\alpha_1+\alpha_2,N}, s_{\alpha_1}s_{\alpha_2}) = \delta_{\alpha_1}^N \varepsilon(H_{\alpha_2,N}, s_{\alpha_1}s_{\alpha_2}s_{\alpha_1}) = \delta_{\alpha_1}^N \varepsilon(H_{\alpha_2,N}, s_{\alpha_2}s_{\alpha_1}s_{\alpha_2}).$$

Writing  $\alpha > 0$  on  $s\mathfrak{C}_o$  to mean that  $\alpha$  hyperplanes are positive on  $s\mathfrak{C}_o$ , we have

$$\varepsilon(H_{\alpha_1+\alpha_2,N}, s) = \begin{cases} \delta_{\alpha_1}^N \varepsilon(H_{\alpha_2,N}, s_{\alpha_1}s) & \text{if } \alpha_1 < 0, \alpha_2 > 0 \text{ on } s\mathfrak{C}_o, \\ -\delta_{\alpha_1}^N \varepsilon(H_{\alpha_2,N}, s) & \text{if } \alpha_1 > 0, \alpha_2 > 0 \text{ on } s\mathfrak{C}_o, \\ \delta_{\alpha_1}^N \varepsilon(H_{\alpha_2,N}, s_{\alpha_2}s) & \text{if } \alpha_1 > 0, \alpha_2 < 0 \text{ on } s\mathfrak{C}_o. \end{cases}$$

Note that  $\alpha_1 + \alpha_2$  hyperplanes are positive on  $s\mathfrak{C}_o$  if and only if  $\alpha_2 = s_{\alpha_1}(\alpha_1 + \alpha_2)$  hyperplanes are positive on  $s_{\alpha_1}s\mathfrak{C}_o$ . By Remark 5.2.3, we may rewrite the previous equation as:

$$\varepsilon(H_{\alpha_1+\alpha_2,N}, s) = \begin{cases} -\delta_{\alpha_1}^N \varepsilon(H_{\alpha_2,N}, s_{\alpha_1}s) & \text{if } \alpha_1, \alpha_2 > 0 \text{ on } s\mathfrak{C}_o, \\ \delta_{\alpha_1}^N \varepsilon(H_{\alpha_2,N}, s_{\alpha_1}s) & \text{otherwise.} \end{cases}$$

In the case where  $|\gamma| = |\alpha|$ , the root subsystem generated by  $\alpha$  and  $\gamma$  is type  $A_2$  as  $(\gamma, \alpha) \neq 0$ . We assign  $\alpha_1 + \alpha_2 = \gamma$  and  $\alpha_1 = \alpha$ , without loss of generality. The first formula in the Proposition now follows from our initial remarks in the proof of this Proposition.

**Case  $|\alpha|^2 = 2|\gamma|^2$ :** Again, we consider the rank 2 case using the setup of Figure 5-1. The roots  $\gamma$  and  $\alpha$  generate a root system of type  $B_2$  and they must correspond to  $\alpha_1 + \alpha_2$  and  $\alpha_1$ , respectively. Our equation from Proposition 5.2.1 for  $m = 0$  gives:

$$\begin{array}{ccccccc}
- & + & & + & - & & - & + & + & - \\
\varepsilon(C_1, C_2) & + & \varepsilon(C_5, C_6) & + & 2\varepsilon(C_3, C_4)\varepsilon(\bar{r}_4 C_6, \bar{r}_4 C_7) & = & 0. \\
H_{\alpha_1+\alpha_2, 2m+n} & & H_{\alpha_1+\alpha_2, 2m+n} & & H_{\alpha_2, n} & & H_{\alpha_1, m+n}
\end{array}$$

As  $\varepsilon = \pm 1$ , letting  $n = N$ , we can rewrite this equation as

$$\begin{aligned}\varepsilon(H_{\alpha_1+\alpha_2,N}, s_{\alpha_2}s_{\alpha_1}s_{\alpha_2}) &= -\varepsilon(H_{\alpha_1+\alpha_2,N}, s_{\alpha_1}s_{\alpha_2}s_{\alpha_1}s_{\alpha_2}) \\ &= \varepsilon(H_{\alpha_2,N}, s_{\alpha_1}s_{\alpha_2}s_{\alpha_1}s_{\alpha_2})\varepsilon(H_{\alpha_1,N}, s_{\alpha_2}s_{\alpha_1}) \\ &= \delta_{\alpha_1}^N \varepsilon(H_{\alpha_2,N}, s_{\alpha_1}s_{\alpha_2}s_{\alpha_1}s_{\alpha_2}) \quad \text{by Lemma 5.3.2.}\end{aligned}$$

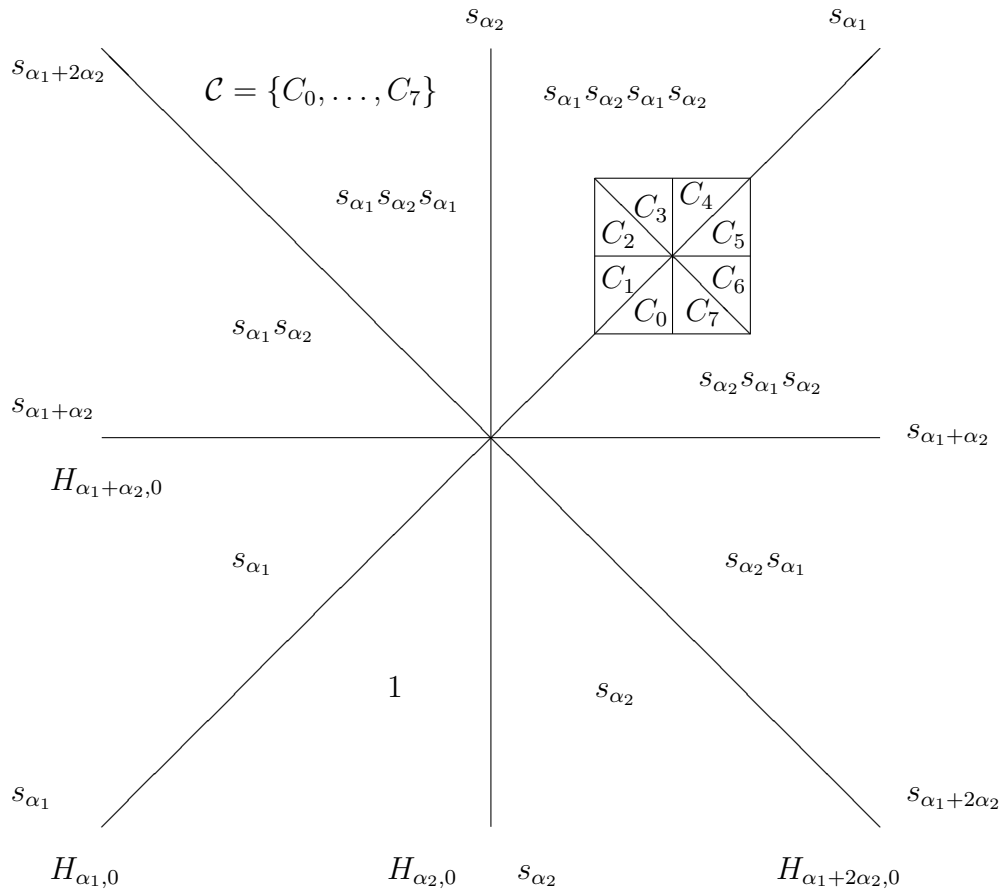


Figure 5-5: Type  $B_2$ : calculating  $\varepsilon$  for  $H_{\alpha_1+\alpha_2,N}$

Proposition 5.2.1 indicates that  $\varepsilon(H_{\alpha_1+\alpha_2,N}, s)$  changes sign as we cross the hyperplanes  $H_{\alpha_1,0}$  and  $H_{\alpha_1+2\alpha_2,0}$ , so we also have

$$\varepsilon(H_{\alpha_1+\alpha_2,N}, s_{\alpha_1}s_{\alpha_2}) = -\varepsilon(H_{\alpha_1+\alpha_2,N}, s_{\alpha_1}s_{\alpha_2}s_{\alpha_1}) = \delta_{\alpha_1}^N \varepsilon(H_{\alpha_2,N}, s_{\alpha_1}s_{\alpha_2}s_{\alpha_1}s_{\alpha_2}).$$

Combining these equations and noting that  $\alpha_1 + \alpha_2$  hyperplanes are positive on  $s\mathfrak{C}_o$  if and only if  $\alpha_2 = s_{\alpha_1}(\alpha_1 + \alpha_2)$  hyperplanes are positive on  $s_{\alpha_1}s\mathfrak{C}_o$ , we have

$$\varepsilon(H_{\alpha_1+\alpha_2,N}, s) = \begin{cases} -\delta_{\alpha_1}^N \varepsilon(H_{\alpha_2,N}, s_{\alpha_1}s) & \text{if } \alpha_1 > 0, \alpha_1 + 2\alpha_2 > 0 \text{ on } s\mathfrak{C}_o \\ \delta_{\alpha_1}^N \varepsilon(H_{\alpha_2,N}, s_{\alpha_1}s) & \text{if } \alpha_1 < 0 \text{ or } \alpha_1 + 2\alpha_2 < 0 \text{ on } s_{\alpha_1}s\mathfrak{C}_o. \end{cases}$$

As before, the second equation of the Proposition now follows from our initial remarks.

**Case  $|\gamma|^2 = 2|\alpha|^2$ :** Again, we consider the rank 2 case using the setup of Figure 5-1. The roots  $\gamma$  and  $\alpha$  generate a root system of type  $B_2$  and they must correspond to  $\alpha_1 + 2\alpha_2$  and  $\alpha_2$ , respectively. Our equation from Proposition 5.2.1 for  $n = 0$  gives:

$$\begin{array}{ccccccc} - & + & & + & - & & - & + & & + & - \\ \varepsilon(C_2, C_3) & + & \varepsilon(C_6, C_7) & + & 2\varepsilon(C_0, C_1)\varepsilon(\bar{r}_1 C_1, \bar{r}_1 C_2) & = & 0. \\ H_{\alpha_1+2\alpha_2, m+n} & & H_{\alpha_1+\alpha_2, m+n} & & H_{\alpha_1, m} & & H_{\alpha_2, 2m+n} \end{array}$$

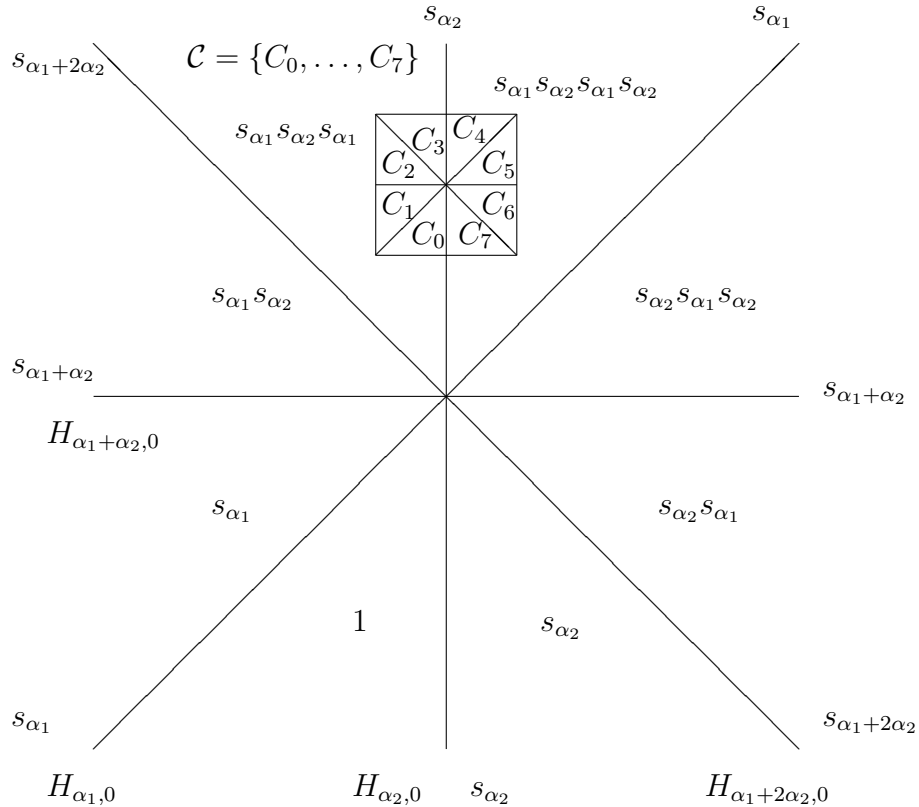


Figure 5-6: Type  $B_2$ : calculating  $\varepsilon$  for  $H_{\alpha_1+2\alpha_2,N}$

As  $\varepsilon = \pm 1$ , letting  $m = N$ , we can rewrite this equation as

$$\begin{aligned}
\varepsilon(H_{\alpha_1+2\alpha_2,N}, s_{\alpha_1}s_{\alpha_2}s_{\alpha_1}s_{\alpha_2}) &= -\varepsilon(H_{\alpha_1+2\alpha_2,N}, s_{\alpha_1}s_{\alpha_2}s_{\alpha_1}) \\
&= \varepsilon(H_{\alpha_1,N}, s_{\alpha_1}s_{\alpha_2}s_{\alpha_1})\varepsilon(H_{\alpha_2,2N}, s_{\alpha_2}s_{\alpha_1}) \\
&= \varepsilon(H_{\alpha_1,N}, s_{\alpha_1}s_{\alpha_2}s_{\alpha_1}) \quad \text{by Lemma 5.3.2.}
\end{aligned}$$

Proposition 5.2.1 indicates that  $\varepsilon(H_{\alpha_1+2\alpha_2,N}, s)$  changes sign as we cross the hyperplanes  $H_{\alpha_2,0}$  and  $H_{\alpha_1+\alpha_2,0}$ , so we also have

$$\varepsilon(H_{\alpha_1+2\alpha_2,N}, s_{\alpha_2}s_{\alpha_1}) = -\varepsilon(H_{\alpha_1+2\alpha_2,N}, s_{\alpha_2}s_{\alpha_1}s_{\alpha_2}) = -\varepsilon(H_{\alpha_1,N}, s_{\alpha_1}s_{\alpha_2}s_{\alpha_1}).$$

Combining these equations and noting that  $\alpha_1 + 2\alpha_2$  hyperplanes are positive on  $s\mathfrak{C}_o$  if and only if  $\alpha_1 = s_{\alpha_2}(\alpha_1 + 2\alpha_2)$  hyperplanes are positive on  $s_{\alpha_2}s\mathfrak{C}_o$ , we have

$$\varepsilon(H_{\alpha_1+\alpha_2,N}, s) = \begin{cases} \varepsilon(H_{\alpha_1,N}, s_{\alpha_2}s) & \text{if } \alpha_2 > 0, \alpha_1 + \alpha_2 > 0 \text{ on } s\mathfrak{C}_o, \\ -\varepsilon(H_{\alpha_1,N}, s_{\alpha_2}s) & \text{if } \alpha_2 < 0 \text{ or } \alpha_1 + \alpha_2 < 0 \text{ on } s\mathfrak{C}_o. \end{cases}$$

As before, the third equation of the Proposition now follows from our initial remarks.

□

**Lemma 5.3.5.** *For a positive root  $\alpha$  and  $\beta = s_{i_1} \cdots s_{i_k} \alpha$  where  $\text{ht}(s_{i_j} \cdots s_{i_k} \alpha) > \text{ht}(s_{i_{j+1}} \cdots s_{i_k} \alpha)$  for  $j = 1, \dots, k$ ,  $s_{i_1} \cdots s_{i_k}$  is a reduced expression.*

*Proof.* Assume, by contradiction, that  $s_{i_1} \cdots s_{i_k}$  is not reduced. By the deletion condition (see [4], Theorem 1.7), there are indices  $j_1 < j_2$  such that  $s_{i_1} \cdots s_{i_k} = s_{i_1} \cdots s_{i_{j_1}} \cdots s_{i_{j_2}} \cdots s_{i_k}$ . It suffices to consider the case where  $j_1 = 1$  and  $j_2 = k$ . Again, from the deletion condition,  $\alpha_{i_1} = s_{i_2} \cdots s_{i_{k-1}} \alpha_{i_k}$ . Now  $(\alpha_{i_1}, \beta) = (s_{i_2} \cdots s_{i_{k-1}} \alpha_{i_k}, s_{i_1} \cdots s_{i_k} \alpha) = (s_{i_2} \cdots s_{i_{k-1}} \alpha_{i_k}, s_{i_2} \cdots s_{i_{k-1}} \alpha) = (\alpha_{i_k}, \alpha)$ . Since applying  $s_{i_1}$  to  $\beta$  decreases the height while applying  $s_{i_k}$  to  $\alpha$  increases the height, therefore  $(\alpha_{i_1}, \beta) > 0$  while  $(\alpha_{i_k}, \alpha) < 0$ , which gives us a contradiction. □

**Theorem 5.3.6.** *Let  $\gamma$  be a positive root, and let  $\gamma = s_{i_1} \cdots s_{i_{k-1}} \alpha_{i_k}$  be such that  $\text{ht}(s_{i_j} \cdots s_{i_{k-1}} \alpha_{i_k})$*

decreases as  $j$  increases. Let  $w_\gamma = s_{i_1} \cdots s_{i_k}$ . If  $\gamma$  hyperplanes are positive on  $s\mathfrak{C}_0$ , then

$$\begin{aligned} \varepsilon(H_{\gamma,N}, s) &= (-1)^{N\#\{\text{noncompact } \alpha_{i_j} : |\alpha_{i_j}| \geq |\gamma|\}} \\ &\times (-1)^{\#\{\beta \in \Delta(w_\gamma^{-1}) : |\beta| = |\gamma| \text{ and } \beta, s_\beta \gamma \in \Delta(s^{-1})\}} \\ &\times (-1)^{\#\{\beta \in \Delta(w_\gamma^{-1}) : |\beta|^2 = 2|\gamma|^2 \text{ and } \beta, -s_\beta s_\gamma \beta \in \Delta(s^{-1})\}} \\ &\times (-1)^{\#\{\beta \in \Delta(w_\gamma^{-1}) : 2|\beta|^2 = |\gamma|^2 \text{ and neither } \beta \text{ nor } -s_\beta s_\gamma \beta \text{ belong to } \Delta(s^{-1})\}}. \end{aligned}$$

*Proof.* Note that  $s_{i_1} \cdots s_{i_{k-1}}$  must be reduced, by Lemma 5.3.5. Combined with the fact that  $\gamma = s_{i_1} \cdots s_{i_{k-1}} \alpha_{i_k} > 0$ , we deduce that  $s_{i_1} \cdots s_{i_{k-1}} s_{i_k}$  must also be reduced by Lemma 1.6 of [4]. By (5.2.1),  $\Delta(w_\gamma^{-1}) = \{\alpha_{i_1}, s_{i_1} \alpha_{i_2}, \dots, s_{i_1} \cdots s_{i_{k-1}} \alpha_{i_k}\}$ .

Let  $w_j = (s_{i_1} \cdots s_{i_j})^{-1} s \in W$  and  $\gamma_j = (s_{i_1} \cdots s_{i_j})^{-1} \gamma$  for  $j = 0, 1, \dots, k-1$ . Note that  $w_0 = s$  and  $\gamma_0 = \gamma$ . Also,  $w_j = s_{i_j} w_{j-1}$  and  $\gamma_j = s_{i_j} \gamma_{j-1}$ . Observe that  $\gamma$  is positive on  $s\mathfrak{C}_0$  if and only if  $\gamma_j$  is positive on  $w_j \mathfrak{C}_0$ . As  $\text{ht}(\gamma_j) > \text{ht}(\gamma_{j+1})$ , therefore  $(\gamma_j, \alpha_{i_{j+1}}) > 0$ . Thus by Proposition 5.3.4,

If  $|\gamma_j| = |\alpha_{i_{j+1}}|$ :

$$\varepsilon(H_{\gamma_j,N}, w_j) = \begin{cases} -\delta_{\alpha_{i_{j+1}}}^N \varepsilon(H_{\gamma_{j+1},N}, w_{j+1}) & \text{if } \alpha_{i_{j+1}} > 0 \text{ and } \gamma_{j+1} > 0 \text{ on } w_j \mathfrak{C}_0, \\ \delta_{\alpha_{i_{j+1}}}^N \varepsilon(H_{\gamma_{j+1},N}, w_{j+1}) & \text{otherwise.} \end{cases}$$

If  $2|\gamma_j|^2 = |\alpha_{i_{j+1}}|^2$ :

$$\varepsilon(H_{\gamma_j,N}, w_j) = \begin{cases} -\delta_{\alpha_{i_{j+1}}}^N \varepsilon(H_{\gamma_{j+1},N}, w_{j+1}) & \text{if } \alpha_{i_{j+1}}, \alpha_{i_{j+1}} + 2\gamma_{j+1} = s_{\gamma_{j+1}} \alpha_{i_{j+1}} > 0 \text{ on } w_j \mathfrak{C}_0, \\ \delta_{\alpha_{i_{j+1}}}^N \varepsilon(H_{\gamma_{j+1},N}, w_{j+1}) & \text{otherwise.} \end{cases}$$

If  $|\gamma_j|^2 = 2|\alpha_{i_{j+1}}|^2$ :

$$\varepsilon(H_{\gamma_j,N}, w_j) = \begin{cases} \varepsilon(H_{\gamma_{j+1},N}, w_{j+1}) & \text{if } \alpha_{i_{j+1}} > 0 \text{ and } \alpha_{i_{j+1}} + \gamma_{j+1} = s_{\gamma_{j+1}} \alpha_{i_{j+1}} > 0 \text{ on } w_j \mathfrak{C}_0, \\ -\varepsilon(H_{\gamma_{j+1},N}, w_{j+1}) & \text{otherwise.} \end{cases}$$

We make the following observations:

1. As the Weyl group preserves length,  $|\gamma_j| = |\gamma|$ .
2. As the Killing form is invariant under the action of the Weyl group, therefore  $\alpha_{i_{j+1}}$  and

$\gamma_{j+1} = (s_{i_1} \cdots s_{i_{j+1}})^{-1} \gamma$  hyperplanes are positive on  $w_j \mathfrak{C}_0 = (s_{i_1} \cdots s_{i_j})^{-1} s \mathfrak{C}_0$  if and only if  $s_{i_1} \cdots s_{i_j} \alpha_{i_{j+1}}$  and  $s_{i_1} \cdots s_{i_j} s_{i_{j+1}} s_{i_j} \cdots s_{i_1} \gamma$  are positive on  $s \mathfrak{C}_0$ .

3. The reflection corresponding to  $s_{i_1} \cdots s_{i_j} \alpha_{i_{j+1}}$  is  $s_{i_1} \cdots s_{i_j} s_{i_{j+1}} s_{i_j} \cdots s_{i_1}$ .

4. Using Proposition 1.2 of [4], since  $\gamma_{j+1} = (s_{i_1} \cdots s_{i_j} s_{i_{j+1}})^{-1} \gamma$ , therefore

$$s_{\gamma_{j+1}} = s_{i_{j+1}} \cdots s_{i_1} s_{\gamma} s_{i_1} \cdots s_{i_{j+1}}.$$

5. From our previous observation, we may conclude that

$$s_{i_1} \cdots s_{i_j} s_{\gamma_{j+1}} \alpha_{i_{j+1}} = -(s_{i_1} \cdots s_{i_j} s_{i_{j+1}} s_{i_j} \cdots s_{i_1}) s_{\gamma} (s_{i_1} \cdots s_{i_j} \alpha_{i_{j+1}}).$$

From these observations,  $k-1$  applications of our equations above and an application of Lemma 5.3.2 give the desired result. □

## Chapter 6

# Extending to non-compact $\mathfrak{h}$

According to our results from Chapter 3, in some sense, the only reducibility hyperplanes we should worry about in computing the signature character are those corresponding to imaginary roots.

Let  $\Delta_i(\mathfrak{g}, \mathfrak{h})$  be the imaginary roots in  $\Delta(\mathfrak{g}, \mathfrak{h})$ . We observe that it satisfies the axioms of a root system, hence it is a semisimple subroot system of  $\Delta(\mathfrak{g}, \mathfrak{h})$ . Let  $\Delta_i^+(\mathfrak{g}, \mathfrak{h})$  be the intersection of  $\Delta_i(\mathfrak{g}, \mathfrak{h})$  with  $\Delta^+(\mathfrak{g}, \mathfrak{h})$ . Observe that if we replace  $W_a$  and  $W$  with the affine Weyl group and Weyl group corresponding to  $\Delta_i(\mathfrak{g}, \mathfrak{h})$  in our arguments in Chapters 4 and 5, our arguments carry through to the non-compact Cartan subalgebra case. The remaining difficulty is to determine the set of simple roots corresponding to  $\Delta_i^+(\mathfrak{g}, \mathfrak{h})$  and to calculate  $\varepsilon$  for hyperplanes corresponding to those simple roots (recall  $\delta_\alpha$ ).

We begin with the observation that as  $\theta\Delta^+(\mathfrak{g}, \mathfrak{h}) = \Delta^+(\mathfrak{g}, \mathfrak{h})$ , if there are complex roots, then  $\theta$  is a non-trivial automorphism of the corresponding Dynkin diagram. The only Dynkin diagrams which have a non-trivial automorphism are those of types  $A_n$ ,  $D_n$ , and  $E_6$ . The vertices of the Dynkin diagram fixed by  $\theta$  correspond to the imaginary simple roots, and the others to the complex simple roots.

Let  $\Pi_i = \{\alpha \in \Pi \mid \alpha \text{ is imaginary}\}$  and  $\Pi_{\mathbb{C}} = \{\alpha \in \Pi \mid \alpha \text{ is complex}\}$ .

**Proposition 6.1.** *The set of simple roots corresponding to  $\Delta_i^+(\mathfrak{g}, \mathfrak{h})$  is*

$$\Pi^i := \Pi_i \cup \{\alpha^i \mid \alpha \in \Pi_{\mathbb{C}}\}$$

where  $\alpha^i$  is defined to be  $\alpha + \alpha_{i_1} + \cdots + \alpha_{i_m} + \theta\alpha$  if the segment of the Dynkin diagram from  $\alpha$  to  $\theta\alpha$

$$is (\alpha) - (\alpha_{i_1}) - \cdots - (\alpha_{i_m}) - (\theta\alpha).$$

*Proof.* It is clear that  $\alpha \in \Pi_i$  is indecomposable as a sum of positive imaginary roots.

Note that  $\alpha, \alpha_{i_1}, \dots, \alpha_{i_m}, \theta\alpha$  all have the same length. Since  $\theta$  flips the segment of the Dynkin diagram  $(\alpha) - (\alpha_{i_1}) - \cdots - (\alpha_{i_m}) - (\theta\alpha)$ , therefore  $\theta\alpha_{i_k} = \alpha_{i_{m+1-k}}$ . From knowledge of type  $A_n$  root systems, we see that  $\alpha^i$  cannot be decomposed into the sum of two positive imaginary roots.

Listing the roots in root systems of types  $A_n$ ,  $D_n$ , and  $E_6$  and possible  $\theta$ , we see that we have found all the roots in  $\Delta_i(\mathfrak{g}, \mathfrak{h})$  which are indecomposable.  $\square$

Now we compute  $\varepsilon$  for  $H_{\alpha^i, n}$  where  $\alpha \in \Pi_{\mathbb{C}}$ . We assume  $\lambda$  to be imaginary and  $(\lambda + \rho, (\alpha^i)^\vee) = n$ .

We may assume that  $\mathfrak{g}$  is type  $A_m$ ,  $\alpha = \alpha_1$ , and the Dynkin diagram is  $(\alpha_1) - (\alpha_2) - \cdots - (\alpha_m)$  so that  $\alpha^i = \alpha_1 + \alpha_2 + \cdots + \alpha_m$ .

Recall the definition of  $X_{\alpha_j}, Y_{\alpha_j}$ , and  $H_{\alpha_j}$  from Lemma 5.3.2. The definition is unique up to multiplication of  $X_{\alpha_j}$  by  $c$  and  $Y_{\alpha_j}$  by  $c^{-1}$  for some non-zero scalar  $c$ . In the case where  $\alpha_j$  is complex, complex conjugation preserves  $\mathfrak{g}_{\alpha_j} + \mathfrak{g}_{-\alpha_j} + \mathfrak{g}_{\theta\alpha_j} + \mathfrak{g}_{-\theta\alpha_j}$ . We have  $\bar{\mathfrak{g}}_{\alpha_j} = \mathfrak{g}_{-\theta\alpha_j}$  and  $\bar{\mathfrak{g}}_{-\alpha_j} = \mathfrak{g}_{\theta\alpha_j}$ . We may choose  $c$  so that

$$-\bar{Y}_{\alpha_j} = X_{\theta\alpha_j} \quad \text{and} \quad -\bar{Y}_{\theta\alpha_j} = X_{\alpha_j} \quad \text{when } j \neq \frac{m+1}{2}.$$

In order to compute  $\varepsilon$ , we use concepts introduced in [9]. Let  $g_1, \dots, g_m \in \mathfrak{g}$  be linearly independent. In [9], the authors give meaning to some monomials

$$g_{i_1}^{\gamma_1} \cdots g_{i_N}^{\gamma_N}, \tag{6.1}$$

where the  $\gamma_j$  are complex numbers by associating them with appropriate elements of the universal enveloping algebra. Let  $J_u = \{1 \leq j \leq m | i_j = u\}$  and let  $\gamma^u = \sum_{j \in J_u} \gamma_j$ . In the case where  $\gamma_1, \dots, \gamma_N$  are non-negative integers, by using appropriate commutation relations, we have

$$g_{i_1}^{\gamma_1} \cdots g_{i_N}^{\gamma_N} = \sum_{j_1, \dots, j_m=0}^{\infty} P_{j_1 \dots j_m}(g_1, \dots, g_m) g_1^{\gamma^1 - j_1} \cdots g_m^{\gamma^m - j_m} \tag{6.2}$$

for some elements  $P_{j_1 \dots j_m}(g_1, \dots, g_m)$  of  $U([\mathfrak{g}, \mathfrak{g}]) \subset U(\mathfrak{g})$ . The  $P_{j_1 \dots j_m}(g_1, \dots, g_m)$  are polynomial in the  $\gamma_j$ , and thus we may extend the  $P_{j_1 \dots j_m}$  to all possible  $\gamma_j$  and not just non-negative integral  $\gamma_j$ .



**Definition 6.2.** *If the following conditions are satisfied:*

1. *All  $\gamma^u$  are non-negative integers.*

2. *If  $j_u > \gamma^u$ , then  $P_{j_1 \dots j_m}(g_1, \dots, g_m) = 0$*

*then the monomial (6.1) is said to **make sense**. If the monomial makes sense, then the right side of equation (6.2) is an element of  $U(\mathfrak{g})$  and we may say that (6.1) is equal to it.*

Given  $w = s_{i_N} \cdots s_{i_1} \in W$  and  $\lambda \in \mathfrak{h}^*$ , we define  $\lambda_0, \lambda_1, \dots, \lambda_N \in \mathfrak{h}^*$  by:

$$\lambda_j + \rho = s_{i_j} \cdots s_{i_1}(\lambda + \rho).$$

As  $s_\beta \mu - \mu$  is a multiple of  $\beta$  for any  $\beta, \mu \in \mathfrak{h}^*$ , we may define the scalars  $\gamma_j$  for  $1 \leq j \leq N$  so that  $\lambda_j - \lambda_{j-1} = \gamma_j \alpha_{i_j}$ .

**Definition 6.3.** *Using the notation defined above and letting  $Y_j = Y_{\alpha_j}$ , we define*

$$F(w; \lambda) = Y_{i_N}^{-\gamma_N} \cdots Y_{i_1}^{-\gamma_1}.$$

**Lemma 6.4.** *(Malikov-Feigin-Fuks,[9]) If  $F(w; \lambda)$  makes sense, then  $F(w; \lambda)v_\lambda$  is a singular vector of the Verma module  $M(\lambda)$ .*

**Theorem 6.5.** *(Malikov-Feigin-Fuks,[9]) If  $(\lambda + \rho, \alpha^\vee) = n$  where  $\alpha$  is a positive root and  $n$  is a positive integer, then  $F(s_\alpha; \lambda)$  makes sense and  $F(s_\alpha; \lambda)v_\lambda$  is a singular vector of the Verma module  $M(\lambda)$  of weight  $\lambda - n\alpha$ .*

**Remark 6.6.** *Here, we must make the observation that for a complex semisimple Lie algebra viewed as a Kac-Moody algebra, all roots are real roots.*

We return to the setup where  $\alpha^i = \alpha_1 + \cdots + \alpha_m$ . We define

$$\begin{aligned} c_1 &= (\lambda + \rho, \alpha_1^\vee) \\ c_2 - c_1 &= (\lambda + \rho, \alpha_2^\vee) \\ &\vdots \\ c_{m-1} - c_{m-2} &= (\lambda + \rho, \alpha_{m-1}^\vee) \\ n - c_{m-1} &= (\lambda + \rho, \alpha_m^\vee). \end{aligned}$$

If we use  $s_1 s_2 \cdots s_m \cdots s_2 s_1$  as a reduced expression for  $s_{\alpha^i}$ , then

$$F(s_{\alpha^i}; \lambda) = Y_1^{n-c_1} Y_2^{n-c_2} \cdots Y_{m-1}^{n-c_{m-1}} Y_m^n Y_{m-1}^{c_{m-1}} \cdots Y_2^{c_2} Y_1^{c_1}.$$

If we use  $s_m s_{m-1} \cdots s_1 \cdots s_{m-1} s_m$  as a reduced expression for  $s_{\alpha^i}$  instead, we get

$$\begin{aligned} F(s_{\alpha^i}; \lambda) &= Y_m^{n-(n-c_{m-1})} Y_{m-1}^{n-(n-c_{m-2})} \cdots Y_2^{n-(n-c_1)} Y_1^n Y_2^{n-c_1} \cdots Y_{m-1}^{n-c_{m-2}} Y_m^{n-c_{m-1}} \\ &= Y_m^{c_{m-1}} Y_{m-1}^{c_{m-2}} \cdots Y_2^{c_1} Y_1^n Y_2^{n-c_1} \cdots Y_{m-1}^{n-c_{m-2}} Y_m^{n-c_{m-1}} \end{aligned}$$

**Lemma 6.7.** *In our setup above,*

$$F(s_1 s_2 \cdots s_m \cdots s_2 s_1; \lambda) = F(s_m s_{m-1} \cdots s_1 \cdots s_{m-1} s_m; \lambda).$$

*Proof.* We prove this by induction on  $m$ . If  $m = 1$ , this is clear. If  $m = 2$ :

$$\begin{aligned} Y_1^{n-c_1} Y_2^n Y_1^{c_1} &= Y_1^{n-c_1} \sum_j \binom{c_1}{j} \binom{n}{j} Y_1^{c_1-j} Y_2^{n-j} [Y_2, Y_1]^j \\ &= \sum_j \binom{c_1}{j} \binom{n}{j} Y_1^{n-j} Y_2^{n-j} [Y_2, Y_1]^j \\ &= \sum_j \binom{c_1}{j} \binom{n}{j} Y_1^{n-j} Y_2^{c_1-j} [Y_2, Y_1]^j Y_2^{n-c_1} \\ &= Y_2^{c_1} Y_1^n Y_2^{n-c_1} \end{aligned}$$

by Proposition 2.2 (2) of [9]. Now assume  $m > 2$ . Let  $\lambda' = s_1(\lambda + \rho) - \rho$  and  $\alpha' = \alpha_2 + \alpha_3 + \cdots + \alpha_m$ .

Then

$$\begin{aligned} c_2 &= (\lambda + \rho, \alpha_2^\vee) \\ c_3 - c_2 &= (\lambda + \rho, \alpha_3^\vee) \\ &\vdots \\ c_{m-1} - c_{m-2} &= (\lambda + \rho, \alpha_{m-1}^\vee) \\ n - c_{m-1} &= (\lambda + \rho, \alpha_m^\vee). \end{aligned}$$

Note that  $(\lambda' + \rho, (\alpha')^\vee) = n$ . Applying our induction hypothesis for  $m - 1$  to  $\alpha'$  and  $\gamma'$ ,

$$\begin{aligned} F(s_{\alpha'}, \gamma') &= Y_2^{n-c_2} Y_3^{n-c_3} \dots Y_{m-1}^{n-c_{m-1}} Y_m^n Y_{m-1}^{c_{m-1}} \dots Y_3^{c_3} Y_2^{c_2} \\ &= Y_m^{c_{m-1}} Y_{m-1}^{c_{m-2}} \dots Y_3^{c_2} Y_2^n Y_3^{n-c_2} \dots Y_{m-1}^{n-c_{m-2}} Y_m^{n-c_{m-1}}. \end{aligned}$$

Thus, using our knowledge of type  $A_m$ ,

$$\begin{aligned} F(s_1 \dots s_m \dots s_1; \lambda) &= Y_1^{n-c_1} Y_2^{n-c_2} \dots Y_{m-1}^{n-c_{m-1}} Y_m^n Y_{m-1}^{c_{m-1}} \dots Y_2^{c_2} Y_1^{c_1} \\ &= Y_1^{n-c_1} Y_m^{c_{m-1}} Y_{m-1}^{c_{m-2}} \dots Y_3^{c_2} Y_2^n Y_3^{n-c_2} \dots Y_{m-1}^{n-c_{m-2}} Y_m^{n-c_{m-1}} Y_1^{c_1} \\ &= Y_m^{c_{m-1}} Y_{m-1}^{c_{m-2}} \dots Y_3^{c_2} (Y_1^{n-c_1} Y_2^n Y_1^{c_1}) Y_3^{n-c_2} \dots Y_{m-1}^{n-c_{m-2}} Y_m^{n-c_{m-1}} \\ &= Y_m^{c_{m-1}} Y_{m-1}^{c_{m-2}} \dots Y_2^{c_1} Y_1^n Y_2^{n-c_1} \dots Y_{m-1}^{n-c_{m-2}} Y_m^{n-c_{m-1}} \\ &= F(s_m s_{m-1} \dots s_1 \dots s_{m-1} s_m; \lambda). \end{aligned}$$

□

We wish to compute  $(F(s_{\alpha^i}; \lambda))^* F(s_{\alpha^i}; \lambda)$ . Recall that  $-\bar{Y}_j = X_{m+1-j}$  and  $n - c_{m-j} = c_j$ . Using our second expression for  $F(s_{\alpha^i}; \lambda)$  for the left factor and our first expression for the right factor in our setting where we have complex simple roots, we see that the element of the universal enveloping algebra obtained is equal to the element obtained in the case where all of the  $\alpha_j$  are imaginary and compact for  $j \neq \frac{m+1}{2}$ ,  $\alpha_{\frac{m+1}{2}}$  is left unchanged, and we use the first expression for  $F(s_{\alpha^i}; \lambda)$  for both factors. As  $(\alpha_j, \alpha^i) = (\alpha_j, \alpha_1 + \dots + \alpha_m) = 0$  for  $2 \leq j \leq m-1$ , it does no harm to assume that for each of those  $j$ ,  $(\lambda + \rho, \alpha_j^\vee) = (\lambda + \rho, \alpha_{m+1-j}^\vee)$  is a small positive number, so that  $(\lambda + \rho, \alpha_1)$  and  $(\lambda + \rho, \alpha_m)$  are positive.

Take  $\lambda_t$  so that  $\lambda_0 = \lambda$ . We have shown that  $(F(s_{\alpha_1 + \dots + \alpha_m}; \lambda_t))^* F(s_{\alpha_1 + \dots + \alpha_m}; \lambda_t)$  in the case where  $\alpha_1 + \dots + \alpha_m = \alpha^i$  is equal to  $F(s_{\alpha_1 + \dots + \alpha_m}; \lambda_t)^* F(s_{\alpha_1 + \dots + \alpha_m}; \lambda_t)$  in the case where the  $\alpha_j$  for  $j \neq \frac{m+1}{2}$  are chosen to be imaginary and compact instead and  $\alpha_{\frac{m+1}{2}}$  is untouched. Letting  $\epsilon$  be the value of  $\varepsilon(H_{\alpha_1 + \dots + \alpha_m, n}, s)$  where all of the  $\alpha_j$ , even  $\alpha_{\frac{m+1}{2}}$ , are imaginary and compact and all of the  $\alpha_j$  hyperplanes through  $s\mathfrak{C}_0$  are positive,

$$\varepsilon = \begin{cases} \epsilon & \text{if } m \text{ is even,} \\ \delta_{\alpha_{\frac{m+1}{2}}}^n \epsilon & \text{if } m \text{ is odd.} \end{cases}$$

Calculating  $\epsilon$  using Proposition 5.3.4, we obtain the following:

**Proposition 6.8.** *For  $\alpha \in \Pi_{\mathbb{C}}$  for which the segment from  $\alpha$  to  $\theta\alpha$  in the Dynkin diagram has  $m$  vertices,*

$$\varepsilon(H_{\alpha^i, n}) = \begin{cases} (-1)^{m-1} = -1 & \text{if } m \text{ is even,} \\ (-1)^{m-1} \delta_{\alpha_{\frac{m+1}{2}}}^n = \delta_{\alpha_{\frac{m+1}{2}}}^n & \text{if } m \text{ is odd.} \end{cases}$$

**Remark 6.9.** *Here, we observe that we could have arrived at the above answer using Theorem 5.3.6 without adjusting  $\lambda$  so that  $(\lambda + \rho, \alpha_j^\vee) > 0$  for all  $j$  as follows:*

1. Set  $\gamma = \alpha_1 + \alpha_2 + \cdots + \alpha_m$ , and choose  $s$  so that  $\lambda + \rho \in s\mathfrak{C}_0$ . Since  $\gamma = s_1 s_2 \cdots s_{m-1} \alpha_m$ ,  $\Delta(w_\gamma^{-1}) = \{\alpha_1, \alpha_1 + \alpha_2, \dots, \alpha_1 + \cdots + \alpha_m\}$ . Note that  $s_{\alpha_1 + \cdots + \alpha_j} \gamma = \alpha_{j+1} + \cdots + \alpha_m$  and  $(\lambda + \rho, (\alpha_{j+1} + \cdots + \alpha_m)^\vee) = (\lambda + \rho, (\alpha_1 + \cdots + \alpha_{m-j})^\vee)$  by  $\theta$ -invariance of  $(\cdot, \cdot)$ .
2.  $s_{s_{\alpha_1 + \cdots + \alpha_j} \gamma} \gamma = \alpha_1 + \cdots + \alpha_j$ .
3. Thus  $\beta \in \Delta(w_\gamma^{-1})$  such that  $|\beta| = |\gamma|$  and  $\beta, s_\beta \gamma \in \Delta(s^{-1})$  occur in pairs  $\alpha_1 + \cdots + \alpha_j$  and  $\alpha_1 + \cdots + \alpha_{m-j}$ , except for the root  $\alpha_1 + \cdots + \alpha_{m/2}$  which is paired with itself in the case where  $m$  is even. In this case,

$$(\lambda + \rho, (\alpha_1 + \cdots + \alpha_{m/2})^\vee) = \frac{n}{2} > 0.$$

We remark here that  $\beta$  cannot equal  $\gamma$ .

4. Use the formula from Theorem 5.3.6.

Now  $\mathfrak{g}_{\alpha^i} = \mathbb{C}[X_1, [X_2, [\dots [X_{m-1}, X_m]] \cdots]]$  is  $\theta$ -stable as  $\alpha^i$  is imaginary. We have

$$\begin{aligned} \theta[X_1, [X_2, [\dots [X_{m-1}, X_m]] \cdots]] &= [\theta X_1, \theta[X_2, [\dots [X_{m-1}, X_m]] \cdots]] \\ &\vdots \\ &= [\theta X_1, [\theta X_2, [\dots [\theta X_{m-1}, \theta X_m]] \cdots]]. \end{aligned}$$

As  $\theta\alpha_j = \alpha_{m+1-j}$ , we may arrange for  $\theta X_j = X_{m+1-j}$  if  $j \neq \frac{m+1}{2}$ , and  $\theta X_{\frac{m+1}{2}} = \delta_{\alpha_{\frac{m+1}{2}}} X_{\frac{m+1}{2}}$ . Using the Jacobi identity, induction on  $m$ , and type  $A_m$  commutation relations, we may show that

$$[X_m, [X_{m-1}, [\dots [X_2, X_1]] \cdots]] = (-1)^{m-1} [X_1, [X_2, [\dots [X_{m-1}, X_m]] \cdots]].$$

It follows that

$$\theta [X_1, [X_2, [\dots [X_{m-1}, X_m]] \dots]] = \begin{cases} -[X_1, [X_2, [\dots [X_{m-1}, X_m]] \dots]] & \text{if } m \text{ is even,} \\ \delta_{\alpha_{\frac{m+1}{2}}} [X_1, [X_2, [\dots [X_{m-1}, X_m]] \dots]] & \text{if } m \text{ is odd,} \end{cases}$$

whence:

**Lemma 6.10.** *Let  $\alpha$  and  $m$  be as defined in the previous Proposition.  $\alpha^i$  is compact if  $m$  is odd and  $\alpha_{\frac{m+1}{2}}$  is compact, and noncompact otherwise.*

**Theorem 6.11.** *Let  $\alpha \in \Pi_{\mathbb{C}}$ .*

$$\varepsilon(H_{\alpha^i, n}) = \begin{cases} -1 & \text{if } \alpha^i \text{ is noncompact and } m \text{ is even,} \\ (-1)^n & \text{if } \alpha^i \text{ is noncompact and } m \text{ is odd,} \\ 1 & \text{if } m \text{ is compact.} \end{cases}$$

We may adjust Theorem 5.3.6 to obtain an analogous formula for the non-compact Cartan setting. Note that in the case where  $\alpha^i$  is noncompact and  $m$  is even, none of the roots in  $\Delta(\mathfrak{g}, \mathfrak{h})$  are imaginary and the simple roots corresponding to  $\Delta_i^+(\mathfrak{g}, \mathfrak{h})$  are orthogonal to one another. In this case,  $\varepsilon$  is always  $-1$ . In the remaining cases, Theorem 5.3.6 holds if we replace the ambient root system  $\Delta^+(\mathfrak{g}, \mathfrak{h})$  with  $\Delta_i^+(\mathfrak{g}, \mathfrak{h})$ .



# Chapter 7

## Historical context

In this chapter, we will expand on the historical context of the problem solved in this dissertation.

Let  $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$  be a  $\theta$ -stable parabolic subalgebra of  $\mathfrak{g}$  and  $\mathfrak{h} \subset \mathfrak{l}$  be a Cartan subalgebra. Let  $L$  be the normalizer of  $\mathfrak{q}$  in  $G$ . We define  $\rho(\mathfrak{u})$  to be  $\frac{1}{2} \sum_{\alpha \in \Delta(\mathfrak{u}, \mathfrak{h})} \alpha$ . We make these definitions in the context of our setup from previous chapters.

Recall the definition of the production functor,  $\text{pro}_{\mathfrak{q}}^{\mathfrak{g}} : \mathcal{C}(\mathfrak{l}, L \cap K) \rightarrow \mathcal{C}(\mathfrak{g}, L \cap K)$ :

$$\text{pro}_{\mathfrak{q}}^{\mathfrak{g}} V = \text{Hom}_{\mathfrak{q}}(U(\mathfrak{g}), V)_{L \cap K\text{-finite}}.$$

We define  $\mathcal{R}^i : \mathcal{C}(\mathfrak{l}, L \cap K) \rightarrow \mathcal{C}(\mathfrak{g}, K)$  by

$$\mathcal{R}^i V = \Gamma^i \text{pro}_{\mathfrak{q}}^{\mathfrak{g}}(V \otimes \wedge^{\text{top}} \mathfrak{u}).$$

In [11], Vogan conjectured:

**Conjecture 7.1.** *For an irreducible, unitarizable  $(\mathfrak{l}, L \cap K)$ -module  $V$  with infinitesimal character  $\lambda \in \mathfrak{h}^*$ , if*

$$\text{Re}(\alpha, \lambda - \rho(\mathfrak{u})) \geq 0 \quad \forall \alpha \in \Delta(\mathfrak{u}, \mathfrak{h})$$

$$\text{and if } m = \dim \mathfrak{u} \cap \mathfrak{k},$$

*then  $\mathcal{R}^m V$  is also unitarizable.*

In [12], Vogan gave a proof of this conjecture, the fundamental idea of which was to couple the

theory of minimal  $K$ -types with knowledge of a large family of well understood unitary representations which were studied by Harish-Chandra: the tempered unitary representations. The following will describe some work leading up to and inspired by this result.

Important to the study of unitarizability is a duality theorem for cohomological induction functors. In his 1978 IAS lectures, Zuckerman proved what was equivalent to the following duality theorem for the right derived functors  $\Gamma^i$  and  $\Gamma^{2m-i}$ :

**Theorem 7.2.** *For  $0 \leq i \leq 2m$ ,  $X \mapsto \Gamma^i X$  and  $X \mapsto (\Gamma^{2m-i}(X^h))^h$  are naturally equivalent on the subcategory of admissible  $(\mathfrak{k}, \mathfrak{l} \cap \mathfrak{k})$ -modules.*

In [2], Enright and Wallach show that since the forgetful functor is additive, covariant, exact, takes injectives to injectives, and commutes with  $\Gamma$ , one can prove the following (stronger) duality theorem (see Theorem 4.3 in [2]):

**Theorem 7.3.** *If  $X$  is in fact an admissible  $(\mathfrak{g}, \mathfrak{k} \cap \mathfrak{l})$ -module, then the  $\mathfrak{k}$ -module isomorphism*

$$\Gamma^i \mathcal{F} X \simeq (\Gamma^{2m-i}(\mathcal{F} X^h))^h,$$

where  $\mathcal{F}$  denotes the forgetful functor, induces a  $\mathfrak{g}$ -module isomorphism  $\Gamma^i X \simeq (\Gamma^{2m-i}(X^h))^h$ .

We will discuss the implementation of the  $\mathfrak{k}$ -module isomorphism.

For every  $\gamma \in \hat{\mathfrak{k}}$  with corresponding representation  $F_\gamma$ , there is a positive definite  $\mathfrak{k}$ -invariant Hermitian form  $\langle \cdot, \cdot \rangle_\gamma$  on  $F_\gamma$  and its dual pairing on  $F_\gamma^*$ . The pairing of  $\Gamma^i X$  with  $\Gamma^{2m-i}(X^h)$  uses the natural isomorphism

$$\Gamma^i X \simeq \bigoplus_{\gamma \in \hat{\mathfrak{k}}} H^i(\mathfrak{k}, \mathfrak{k} \cap \mathfrak{l}; X \otimes F_\gamma^*) \otimes F_\gamma$$

as  $\mathfrak{k}$ -modules, where the action on the right is on the last term.

We may pair  $H^i(\mathfrak{k}, \mathfrak{k} \cap \mathfrak{l}; X \otimes F_\gamma^*)$  with  $H^{2m-i}(\mathfrak{k}, \mathfrak{k} \cap \mathfrak{l}; X^h \otimes F_\gamma^*)$  by pairing spaces  $C^i$  and  $C^{2m-i}$  in the cochain complexes using the identification

$$C^i(X \otimes F_\gamma^*) = \text{Hom}_{\mathfrak{k} \cap \mathfrak{l}}(\wedge^i(\mathfrak{k}/\mathfrak{k} \cap \mathfrak{l}), X \otimes F_\gamma^*) \simeq [\wedge^i(\mathfrak{k}/\mathfrak{k} \cap \mathfrak{l})]^* \otimes X \otimes F_\gamma^*.$$

Using  $\langle \cdot, \cdot \rangle$ , a Hermitian pairing between  $\wedge^i(\mathfrak{k}/\mathfrak{k} \cap \mathfrak{l})$  and  $\wedge^{2m-i}(\mathfrak{k}/\mathfrak{k} \cap \mathfrak{l})$  defined in section 1 of [13],



we define a  $\mathfrak{k}$ -invariant pairing of  $C^i(\mathfrak{k}, \mathfrak{k} \cap \mathfrak{l}; X \otimes F_\gamma^*)$  and  $C^{2m-i}(\mathfrak{k}, \mathfrak{k} \cap \mathfrak{l}; X^h \otimes F_\gamma^*)$  by

$$\langle \omega_1 \otimes v \otimes f_1^*, \omega_2 \otimes v' \otimes f_2^* \rangle = \langle \omega_1, \omega_2 \rangle \langle v, v' \rangle \langle f_1^*, f_2^* \rangle_\gamma. \quad (7.1)$$

The standard proof of Poincaré duality shows that this gives us a  $\mathfrak{k}$ -invariant pairing at the level of cohomology. We obtain a  $\mathfrak{k}$ -invariant pairing of  $\Gamma^i(X)$  and  $\Gamma^{2m-i}(X^h)$  from the tensor product pairing of  $H^i(\mathfrak{k}, \mathfrak{k} \cap \mathfrak{l}; X \otimes F_\gamma^*) \otimes F_\gamma$  and  $H^{2m-i}(\mathfrak{k}, \mathfrak{k} \cap \mathfrak{l}; X^h \otimes F_\gamma^*) \otimes F_\gamma$ , which induces the  $\mathfrak{g}$ -invariant pairing of the duality theorem of Enright and Wallach.

In the case where  $X$  has an invariant Hermitian form and  $i = m$ , this implies that there is a  $\mathfrak{g}$ -invariant isomorphism  $\Gamma^m X \simeq (\Gamma^m X)^h$ , and so  $\Gamma^m X$  has a  $\mathfrak{g}$ -invariant Hermitian form.

Define  $\mathcal{L}^i : \mathcal{C}(\mathfrak{l}, L \cap K) \rightarrow \mathcal{C}(\mathfrak{g}, K)$  by

$$\mathcal{L}^i V = \Gamma^i \text{ind}_{\mathfrak{q}}^{\mathfrak{g}}(V \otimes \wedge^{\text{top}} \mathfrak{u}).$$

Since  $\text{ind}_{\mathfrak{q}}^{\mathfrak{g}}(V)^h \simeq \text{pro}_{\mathfrak{q}}^g(V^h)$ , if  $V$  is an  $(\mathfrak{l}, L \cap K)$ -module, then  $(\mathcal{L}^i V)^h \simeq \mathcal{R}^{2m-i}(V^h)$ . When studying  $\mathcal{R}^i V$  where  $V$  admits an invariant Hermitian form, we are essentially looking at the duality theorem in the case where  $X$  has an invariant Hermitian form and is the generalized Verma module  $\text{ind}_{\mathfrak{q}}^{\mathfrak{g}}(V)$ .

The general outline of the proof of Conjecture 7.1 in [12] is as follows:

- Any representation which admits an invariant Hermitian form may be obtained from a tempered unitary representation via analytic continuation through representations admitting invariant Hermitian forms.
- Jantzen filtration arguments lead us to conclude that for a  $(\mathfrak{g}, K)$ -module of finite length admitting a non-degenerate invariant Hermitian form  $\langle \cdot, \cdot \rangle$ , there exists some finite collection of tempered irreducible  $(\mathfrak{g}, K)$ -modules  $Z_1, \dots, Z_p$  of formal  $K$ -character  $\Theta(Z_i)$  and integers  $r_1^+, \dots, r_p^+, r_1^-, \dots, r_p^-$  such that the signature of  $\langle \cdot, \cdot \rangle$  is  $\left( \sum_{i=1}^p r_i^+ \Theta(Z_i), \sum_{j=1}^p r_j^- + \Theta(Z_j) \right)$ .
- The tempered characters in this expression for the signature in the case where the  $(\mathfrak{g}, K)$ -module is  $\mathcal{R}V$  must have lowest  $K$ -types in the bottom layer of  $\mathcal{R}V$ . It follows that unitarizability of  $\mathcal{R}V$  is equivalent to the form being definite on the bottom layer.

- Using the ideas of Enright and Wallach (including the construction (7.1) of the invariant Hermitian pairing) described above, one may compare the signatures of the invariant Hermitian forms for  $V$  and  $\mathcal{R}V$  on the bottom layer of  $K$ -types.

Wallach gives an alternate proof to Vogan's conjecture in [13] that does not use the complicated machinery of  $K$ -types and tempered unitary representations.

For  $\lambda \in \mathfrak{z}(\mathfrak{l})$  and  $V$  an admissible, finitely generated  $(\mathfrak{l}, L \cap K)$ -module, Wallach defines  $\mathbb{C}_\lambda$  to be the one-dimensional  $\mathfrak{l}$ -module corresponding to  $\lambda$  and  $V_\lambda$  to be  $\mathbb{C}_\lambda \otimes V$ , which may be extended to a  $\mathfrak{q}$ -module by allowing  $\mathfrak{u}$  to act trivially.  $M(\mathfrak{q}, \lambda, V)$  denotes the generalized Verma module  $U(\mathfrak{g}) \bigotimes_{U(\mathfrak{q})} V_\lambda$ . From irreducible  $V$  which admits an  $\mathfrak{l}$ -invariant Hermitian form  $\langle \cdot, \cdot \rangle$ , one constructs an invariant Hermitian form on  $M(\mathfrak{q}, \lambda, V)$  analogous to the Shapovalov form that we constructed.

Wallach defines  $(V, \lambda, \langle \cdot, \cdot \rangle)$  to be *well placed* if for some  $\xi \in (\mathfrak{z}(\mathfrak{l}) \cap \mathfrak{k})^*$  that is purely imaginary valued on  $\mathfrak{z}(\mathfrak{l}) \cap \mathfrak{k}_0$ ,  $(\xi, \alpha) < 0$  for  $\alpha \in \Delta(\mathfrak{u}, \mathfrak{t})$  and  $M(\mathfrak{q}, \lambda + t\xi, V)$  is irreducible for all  $t \geq 0$ . (Here, we note the connection between the definitions of well placed and Wallach region.)

In the case that  $(V, \lambda, \langle \cdot, \cdot \rangle)$  is well placed,

$$ch_s(M(\mathfrak{q}, \lambda + t\xi, V)) = e^{t\xi} ch_s(M(\mathfrak{q}, \lambda, V)).$$

As discussed previously in Chapter 2, an asymptotic argument as  $t$  goes to infinity gives us a formula for the signature character of  $M(\mathfrak{q}, \lambda, V, \langle \cdot, \cdot \rangle)$  in terms of the signature character of  $(V, \langle \cdot, \cdot \rangle)$ . A similar argument gives us a formula for the character of the generalized Verma module in terms of the character of  $V$ .

As in Vogan's proof, the construction (7.1) of an invariant Hermitian form on  $\Gamma^m X$  is an instrumental component in discussing unitarizability. From the construction, it is clear that

$$ch_s(\Gamma^i X \oplus \Gamma^{2m-i} X) = 0 \tag{7.2}$$

for  $i \neq m$ , whence

$$ch_s \Gamma^i X = ch_s \Gamma^m X. \tag{7.3}$$

Furthermore, the signature character of  $\Gamma^m X$  can be expressed in terms of signatures of the forms

on the  $H^m(\mathfrak{g}, \mathfrak{k}; X \otimes F_\gamma^*)$  and in terms of characters  $chF_\gamma$ :

$$ch_s \Gamma^m X = \sum_{\gamma \in \hat{\mathfrak{k}}} \text{sgn}(H^m(\mathfrak{g}, \mathfrak{k}; X \otimes F_\gamma^*)) chF_\gamma \quad (7.4)$$

where  $\text{sgn}(Y)$  is  $p - q$  if  $(p, q)$  is the signature on  $Y$ .

For  $X = M(\mathfrak{q}, \lambda, V, \langle \rangle)$ , Wallach uses the above equation to calculate the coefficient of  $chF_\gamma$  in  $(-1)^m ch_s \Gamma^m(M(\mathfrak{q}, \lambda, V, \langle \rangle))$  in the case where  $(V, \lambda, \langle \rangle)$  is well placed and  $\langle \rangle$  is positive definite. He then calculates the coefficient of  $chF_\gamma$  in  $\sum_{i=0}^{2m} (-1)^i ch \Gamma^i(M(\mathfrak{q}, \lambda, V, \langle \rangle))$ . The first expression obtained involves  $ch_s M(\mathfrak{q}, \lambda, V, \langle \rangle)$  while the second involves  $ch M(\mathfrak{q}, \lambda, V, \langle \rangle)$ . Manipulating these expressions using the formulas calculated using asymptotic arguments mentioned above, he shows that the two expressions are in fact equal. It follows that

$$(-1)^m ch_s \Gamma^m(M(\mathfrak{q}, \lambda, V, \langle \rangle)) = \sum_{i=0}^{2m} (-1)^i ch \Gamma^i(M(\mathfrak{q}, \lambda, V, \langle \rangle)).$$

Since  $\Gamma^i M(\mathfrak{q}, \lambda, V) = 0$  for  $i \neq m$  and  $(V, \lambda, \langle \rangle)$  well placed, therefore

$$ch_s \Gamma^m(M(\mathfrak{q}, \lambda, V, \langle \rangle)) = ch \Gamma^m(M(\mathfrak{q}, \lambda, V, \langle \rangle)),$$

whence  $\Gamma^m(M(\mathfrak{q}, \lambda, V, \langle \rangle))$  is unitarizable or zero.

Wallach shows that for  $(V, \langle \rangle)$  satisfying the conditions of Conjecture 7.1, with necessary adjustments to accommodate usage of  $\text{ind}_{\mathfrak{q}}^{\mathfrak{g}}$  instead of  $\text{pro}_{\mathfrak{q}}^{\mathfrak{g}}$ ,  $(V, 0, \langle \rangle)$  is well placed. It follows that  $\Gamma^m \text{ind}_{\mathfrak{q}}^{\mathfrak{g}} V$  is unitarizable, proving the conjecture.



## Chapter 8

# Conclusion

As discussed in the introduction, the motivation behind the problem solved in this thesis is the utilization of a formula for the Shapovalov form on an arbitrary irreducible Verma module when it exists in the study of unitarizability of cohomologically induced modules. It is our hope that the techniques used in this thesis involving reducibility hyperplanes may be combined with Wallach's work described in the previous chapter to arrive at an answer to the unitarizability question.

In order to extend the approach of this thesis, we must begin by determining when generalized Verma modules are irreducible. This is an open problem in the most general case. However, we are interested in the case where we are inducing from principal series representations, which could potentially be treated by ideas in [10].

Once we understand reducibility, we may develop formulas analogous to those found in this thesis for  $ch_s M(\mathfrak{q}, \lambda, V, \langle \rangle)$  and for  $ch M(\mathfrak{q}, \lambda, V, \langle \rangle)$  when  $(V, \lambda, \langle \rangle)$  is not necessarily well placed.

We observe that many formulas such as (7.2), (7.3), and (7.4) still hold outside of the Wallach region. Once we have formulas for  $ch_s M(\mathfrak{q}, \lambda, V, \langle \rangle)$  and for  $ch M(\mathfrak{q}, \lambda, V, \langle \rangle)$ , we expect the computations in comparing the signature character of the cohomologically induced module to its character to be analogous to those in [13].

I owe Professor Etingof many thanks for bringing the paper [9] to my attention. Although his original suggestion was to use formulas for singular vectors to calculate  $\varepsilon$ , the suggestion was abandoned due to the complexity of computing projections  $p(x^*x)$ . Nevertheless, the singular vector formulas were useful in handling the non-compact Cartan case in Chapter 6. It may be worthwhile to attempt to draw connections between expressions for singular vectors  $xv_\lambda$ , the value

of  $\lambda(p(x^*x))$ , and the formulas obtained for  $\varepsilon$ , with the goal of determining if shorter more direct proofs for the formulas exist. This in turn may lead to alternate expressions for the sign  $\varepsilon$  which would be more convenient for the computations described above.

# Bibliography

- [1] Jacques Dixmier. *Enveloping Algebras*. Number 11 in Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 1996.
- [2] Thomas Enright and Nolan R. Wallach. Notes on homological algebra and representations of Lie algebras. *Duke Math. J.*, 47(1):1–15, 1980.
- [3] James E. Humphreys. *Introduction to Lie Algebras and Representation Theory*. Number 9 in Graduate Texts in Mathematics. Springer-Verlag, New York, 1972.
- [4] James E. Humphreys. *Reflection groups and Coxeter groups*. Number 29 in Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1990.
- [5] Jens Carsten Jantzen. *Modulen mit einem höchsten Gewicht*. Number 750 in Lecture Notes in Mathematics. Springer, Berlin, 1979.
- [6] Victor G. Kac and David A. Kazhdan. Structure of representations with highest weight of infinite-dimensional Lie algebras. *Advances in Mathematics*, 34(1):97–108, 1979.
- [7] Anthony W. Knap. *Lie Groups Beyond an Introduction*. Number 140 in Progress in Mathematics. Birkhäuser, Boston, 1996.
- [8] Anthony W. Knap and David A. Vogan. *Cohomological induction and unitary representations*. Number 45 in Princeton Mathematical Series. Princeton University Press, Princeton, New Jersey, 1995.
- [9] F.G. Malikov, B.L. Feigin, and D.B. Fuks. Singular vectors in Verma modules over Kac-Moody algebras. *Functional Analysis Applications*, 20(2):103–113, 1986.

- [10] Birgit Speh and David A. Vogan. Reducibility of generalized principal series representations. *Acta Mathematica*, 145(3-4):227–299, 1980.
- [11] David A. Vogan. *Representations of real reductive Lie groups*. Number 15 in Progress in Mathematics. Birkhäuser, Boston, Massachusetts, 1981.
- [12] David A. Vogan. Unitarizability of certain series of representations. *Ann. of Math. (2)*, 120(1):141–187, 1984.
- [13] Nolan R. Wallach. On the unitarizability of derived functor modules. *Invent. Math.*, 78(1):131–141, 1984.